Secondary Magazine
Issue 65

National Centre for Excellence in the Teaching of Mathematics

Welcome to Issue 65 of the Secondary Magazine.

Contents

Contributors to this issue include: Mary Pardoe, Peter Ransom and Jim Thorpe.

From the editor

In this issue of the NCETM Secondary Magazine you will find ideas for the classroom in Focus on… which touches on intriguing aspects of *sense-making* in mathematics. Another article is a sketch by an Open University tutor of some kinds of problem that may arise when students try to apply *rules without reasons*.

The 'roles' in mathematics learning of *sense-making* and *rules without reasons* are essential considerations for all mathematics teachers, and are frequently mentioned in articles by mathematics educators.

For example, in The Mathematical Miseducation Of America's Youth, Michael T. Battista, Professor of Mathematics Education at Ohio State University reminds readers that:
In traditional mathematics instruction, every day is the same: the teacher shows students several examples of how to solve a certain type of problem and then has them practice this method in class and in homework. Instead of understanding what they are doing, students parrot what they have seen and heard.

He points out that in effective mathematics teaching…

increased attention is given to mathematical reasoning and problem solving

…with the focus on…

the basic skills of today, not those of 40 years ago.

Rather than seeing mathematical learning as…

progressing through carefully scripted schedules of acquiring a set of computational skills

…the professor exhorts teachers to remember that…

problem solving, reasoning, justifying ideas, making sense of complex situations, and learning new ideas independently…are now critical skills for all…

…because…

in the Information Age and the web era, obtaining the facts is not the problem; analysing and making sense of them is.

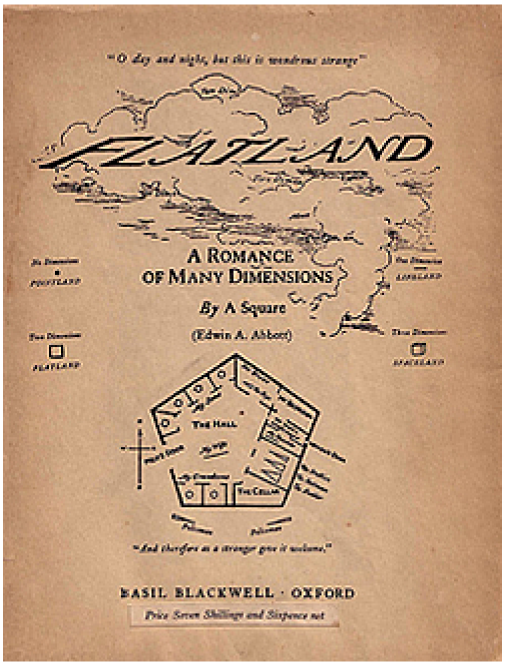
Professor Battista urges that, consequently:
Students' learning of symbolic manipulations must never become disconnected from their reasoning about quantities. For when it does, they become overwhelmed with trying to memorize countless rules for manipulating symbols. Even worse, when students lose sight of what symbol manipulations imply about real-world quantities, doing mathematics becomes an academic ritual that has no real-world usefulness.

Indeed, to be able to use mathematics to make sense of the world, students must first make sense of mathematics.

Mathematical ideas must be personally constructed by students as they try to make sense of situations.

Focus on…Flatland

A Square describes his 'world' in *Flatland* by Edwin Abbott Abbott, second, revised edition, 1884, thus: *Imagine a vast sheet of paper on which straight Lines, Triangles, Squares, Pentagons, Hexagons, and other figures, instead of remaining fixed in their places, move freely about, on or in the surface, but without the power of rising above or sinking below it, very much like shadows--only hard with luminous edges--and you will then have a pretty correct notion of my country and countrymen.*



Edwin A. Abbott (1838 -1926) was a classics graduate of St John's College, Cambridge, who became headmaster of the City of London School, where he himself had been educated.

Although Abbott wrote *Flatland* partly as a satire on the social life and values of Victorian England, this book is one of several fictional works in which trying to imagine existence in a two-dimensional universe can help us make some sense of more than three dimensions.

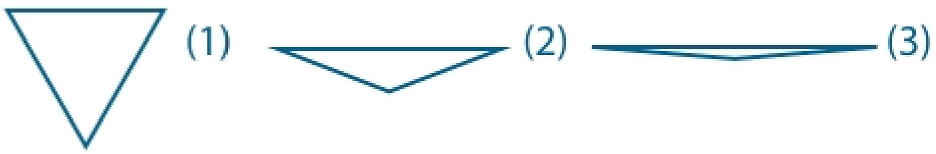
As the copyright has expired, *Flatland* is free to read, listen to, or download. Or you can buy a reprint with an excellent introduction by Thomas Banchoff, a mathematics professor at Brown University.

You can watch a video introduction by Professor Banchoff to Flatland: The Movie. Then you can see an extract from the film, another clip, and a third 'trailer'. The whole film is available for educators to buy as a DVD.

How do Flatlanders appear to each other?
Because Flatlanders are trapped in two-dimensional space, they cannot view each other from above or below. Therefore they appear to one another as straight lines.

This is part of an illustrated explanation by Abbott, who is writing as 'A Square':

Figure 1 represents the Tradesman as you would see him while you were bending over him from above; figures 2 and 3 represent the Tradesman, as you would see him if your eye were close to the level, or all but on the level of the table; and if your eye were quite on the level of the table (and that is how we see him in Flatland) you would see nothing but a straight line.



Flatlanders' sides are illuminated – the nearer a point on a Flatlander's side is to a viewer, the brighter it appears. Every Flatlander, of whatever shape, has just one eye.

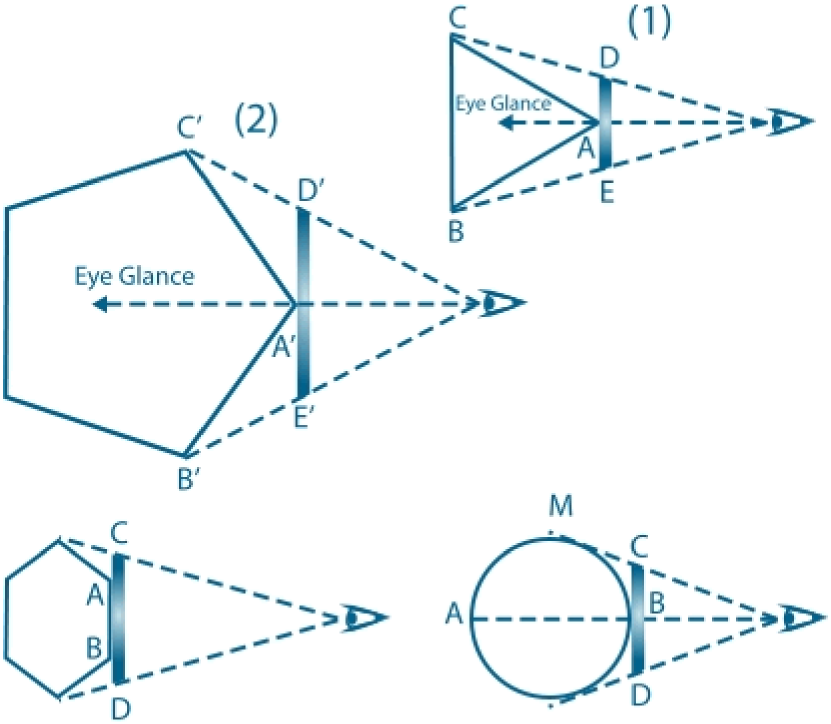
So what do particular Flatlanders look like to each other as they move around in their plane? How do they recognize each other? How do they distinguish between, for example, a square and an equilateral triangle, or between a rectangle and a circle?

Students may enjoy discussing and exploring these questions.

A Square writes:

Suppose I see two individuals approaching whose rank I wish to ascertain. They are, we will suppose, a Merchant and a Physician, or in other words, an Equilateral Triangle and a Pentagon; how am I to distinguish them?

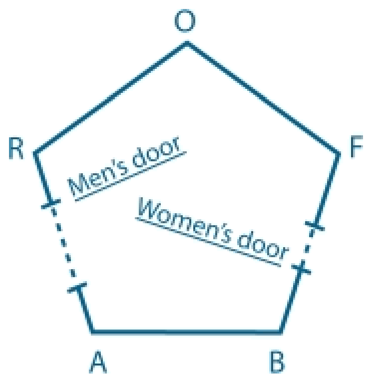
And draws some sketches:



What happens as the eye moves around a polygon?

What are the consequences for Flatlanders of relations between angles and lengths?
A Square explains:

The most common form for the construction of a house is five-sided or pentagonal, as in the annexed figure.



The two Northern sides RO, OF, constitute the roof, and for the most part have no doors; on the East is a small door for the Women; on the West a much larger one for the Men; the South side or floor is usually doorless.

The women's doorway is narrower than the men's doorway because Flatland women are straight lines, whereas all men are polygonal! The higher a man's status – or the social class to which he belongs – the more sides he has. The lowest class consists of soldiers and workmen who are long, 'thin' isosceles triangles with very short 'bases' that are hardly distinguishable from women! But members of the highest class have so many sides that they almost resemble circles.

Assuming that all men, other than those of the lowest class, are regular polygons, what happens as men of various social classes approach, from the inside or from the outside, the corners of buildings that are regular pentagons? Can they touch or 'reach' the corners of buildings – when approaching from the outside?…and from the inside?

What understanding of the idea of an angle is possible for Flatlanders? Are women likely to have a better, or poorer, understanding than men?

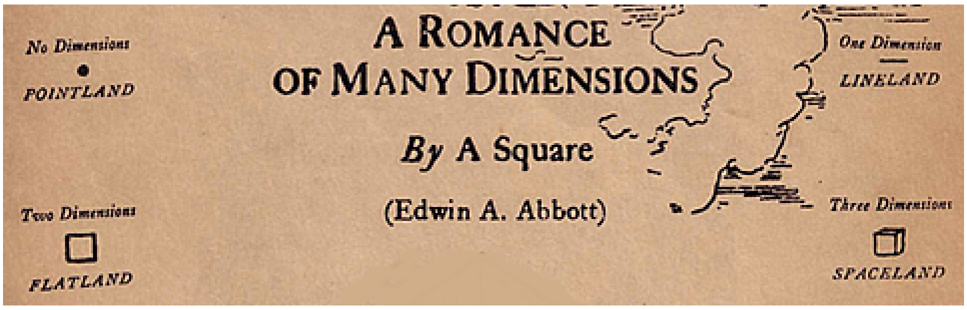
How is the width of the men's doorway related to the side lengths of men of various social classes who can just 'squeeze through'?

Dimensions beyond experience

Abbott tries to help the reader appreciate how beings may be in situations in which it is almost impossible for them to have any conception of, let alone visualise, more dimensions than those of the reality in which they exist.

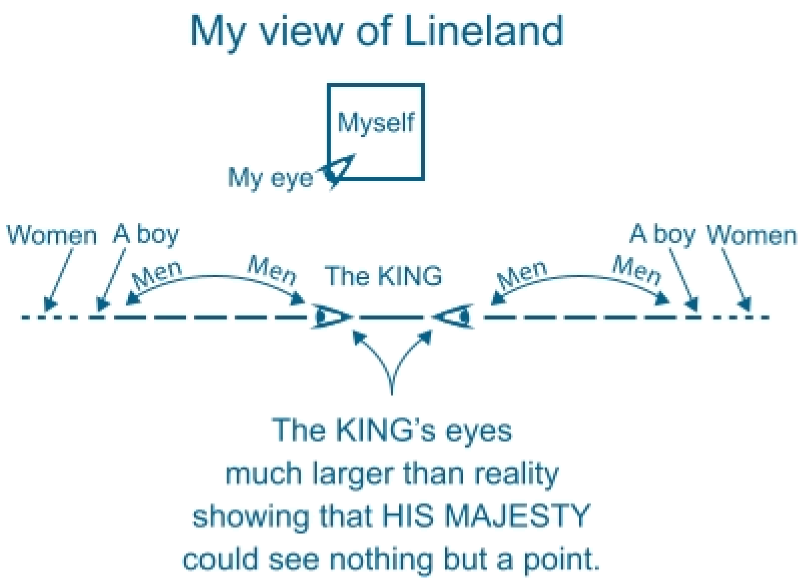
The 'mind-blowing' nature of this incomprehension is the focus in this clip from *Flatland: The Movie*.

'Realities' of 0, 1, 2 and 3 dimensions are represented on the Flatland first edition book wrapper:



A Square dreams about Lineland

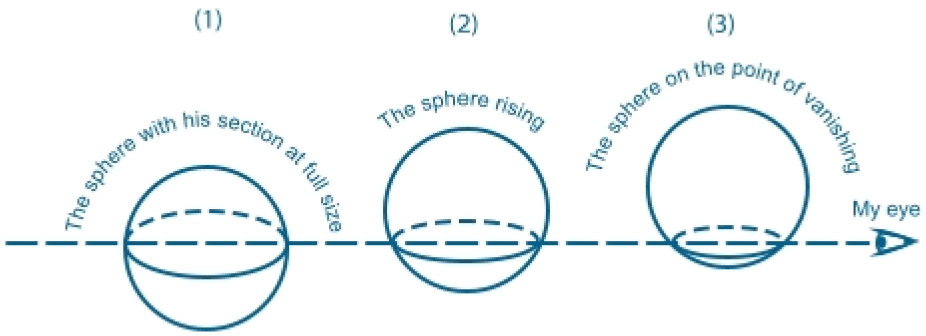
This is a sketch by A Square, of himself seeing, in a dream, some Flatlanders trapped in only one-dimension:



It seemed that this poor ignorant Monarch--as he called himself--was persuaded that the Straight Line which he called his Kingdom, and in which he passed his existence, constituted the whole of the world, and indeed the whole of Space. Not being able either to move or to see, save in his Straight Line, he had no conception of anything out of it. Though he had heard my voice when I first addressed him, the sounds had come to him in a manner so contrary to his experience that he had made no answer, "seeing no man," as he expressed it, "and hearing a voice as it were from my own intestines." Until the moment when I placed my mouth in his World, he had neither seen me, nor heard anything except confused sounds beating against, what I called his side, but what he called his inside or stomach; nor had he even now the least conception of the region from which I had come. Outside his World, or Line, all was a blank to him; nay, not even a blank, for a blank implies Space; say, rather, all was non-existent.

What do Flatlanders see while strangers from Spaceland are passing through their plane?

A Square sketches a sphere, which is a 'stranger from Spaceland', as it passes through the plane that is Flatland…

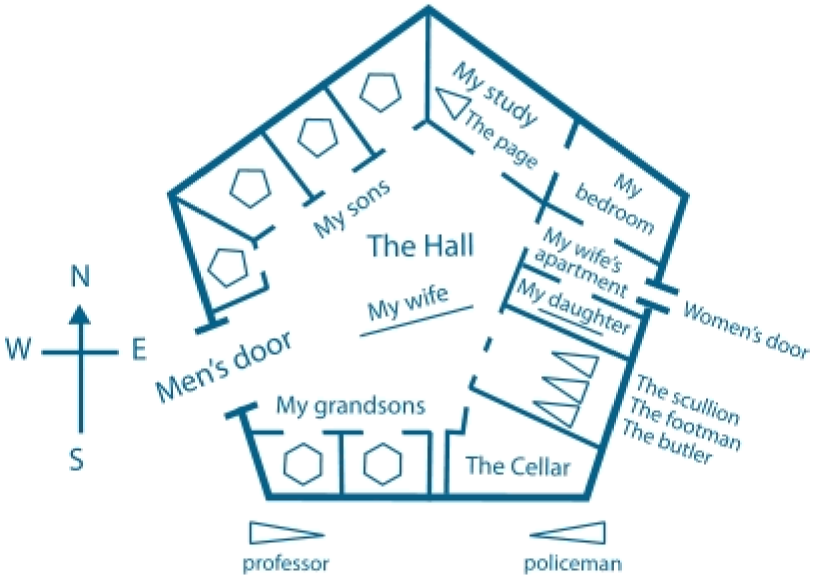


…what would Flatlanders see?

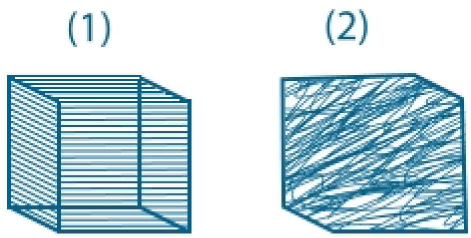
What would Flatlanders see if other strangers from Spaceland, other three-dimensional shapes, were to pass through Flatland?

Escaping into another dimension

The sphere takes A Square out of Flatland into the third dimension, in which A Square sees from above, for the first time, his own house. A Square draws a plan of what he sees:



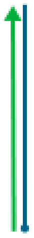
While A Square is in Spaceland, the Sphere shows him how he builds a cube by laying squares one upon another, and A Square draws this sketch:



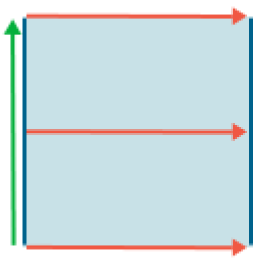
Beings in a three-dimensional universe are able to see how the two-dimensional polygons of Flatland really are, and how they are related to each other. Whereas Flatlanders cannot see polygons properly – they 'see' them as lines. From this it is possible to argue that in order truly and properly to see the real nature of three-dimensional objects, and relations between them, we need to go into a fourth dimension!

We can visualise introducing new dimensions, one at a time, by first imagining a dimensionless point. The imagined point is a zero-dimensional object with no length, width or height:

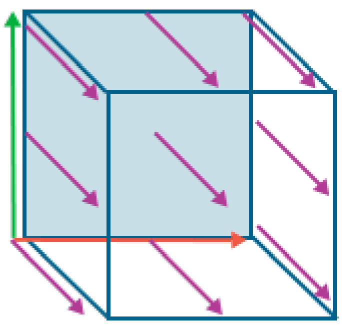
If just one dimension is introduced the point can, in our imagination, be moved in a straight line forwards and backwards, or up and down, but not from side to side – the straight line is a one-dimensional object. We can imagine that the dimensionless point traces a straight-line segment with a length, say one unit, but with no width:



When a second dimension is made available the straight-line segment can be moved one unit in a direction perpendicular to itself so that it 'sweeps out' a square, which is a two-dimensional object:

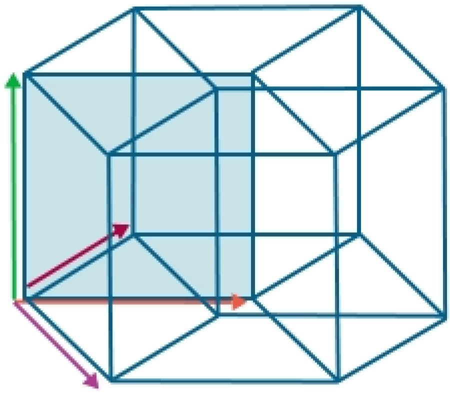


Introducing a third dimension allows the square to be moved one unit in a direction that is perpendicular to the plane in which the square is, and to 'sweep out' a cube, which is a three-dimensional object:



Each step in the train of thinking just described has created an object of which the dimensions are one more than the dimensions of the previous object.

People have tried to visualise the object that is created by taking a fourth 'similar' step. That is, they imagine introducing a fourth dimension, and moving the unit cube one unit in a direction that is perpendicular to all three of the lower dimensions:



The 'mysterious' result, called a tesseract, is difficult to 'see', and to represent in 'our' space.

People make representations in two dimensions of three-dimensional objects, such as cubes, which other people are able to interpret. But it is hard to represent in three dimensions, let alone in only two dimensions, an object in four spatial dimensions!

A *hypercube* is a generalisation of a cube to *n* dimensions. A *one-dimensional hypercube* is a *line segment*, a *two-dimensional hypercube* is a *square*, a *three-dimensional hypercube* is a *cube*, and a *four-dimensional hypercube* is a *tesseract*.

These two-dimensional images represent a dynamic projection of a *tesseract* into three-dimensional space.

You will find some mathematical notation and results related to *hypercubes* at WolframMathWorld.

This video shows representations of rotations of *hypercubes* of dimension 0 up to 6!

Here a transparent *tesseract* rotates about each axis in turn of three mutually perpendicular axes.

This is another two-dimensional representation of a *tesseract* projected into three dimensions.

Factual messages in mathematical fictions

Human beings have great difficulty in 'getting their heads round' the idea of shapes in four spatial dimensions. In *Flatland* Edwin Abbott is suggesting that it may be productive to pursue some ideas that are impossible for us to visualise, and of which it is difficult at first to make any sense.

Other 'mathematical fictions' that use lower-dimensional settings to try to illuminate multi-dimensional ideas include:

- An Episode of Flatland, by Charles Howard Hinton (1907)
- Sphereland: A Fantasy about Curved Spaces and an Expanding Universe, by Dionys Burger (1965)
- The Planiverse: Computer Contact with a Two-dimensional World, by A. K. Dewdney (1984)
- Flatterland: Like Flatland, Only More So, by Ian Stewart (2000)
- Spaceland, by Rudy Rucker (2002).

Algebraic 'rules without reasons'

An Open University tutor uses Richard Skemp's distinction between 'relational' and 'instrumental' understanding to shed some light on why students sometimes make 'howling' errors when responding to algebra 'questions'.

Recent conversations with trainee teachers led me to think again about fluency in algebraic manipulations, and about how pupils make sense of algebra.

In preparation for a recent Open University PGCE Day School, I thought about how I could illuminate the teaching and learning of algebra. As the PGCE students lack experience of the nature of pupils' algebraic malpractices, I thought theoretical pegs might be a good way to hang out distinct but related aspects of algebra, and thereby encourage reflection.

The following remarks build on those conversations with the PGCE student teachers, and represent my attempt to clarify the two distinct but related issues of doing and of understanding algebra.

I settled on Skemp's distinction between 'relational' and 'instrumental' understanding to illuminate the difference between understanding algebraic concepts and merely knowing procedures, such as 'collecting like terms'.

Richard Skemp wrote in *Mathematics Teaching, 77* (pages 20 to 26):

> *It was brought to my attention some years ago by Stieg Mellin-Olsen, of Bergen University, that there are in current use two meanings of the word 'understanding'. These he distinguishes by calling them 'relational understand -ing' and 'instrumental understand -ing'. By the former is meant what I have always meant by understand -ing, and probably most readers of this article: knowing both what to do and why. Instrumental understanding I would until recently not have regarded as understanding at all. It is what I have in the past described as 'rules without reasons', without realising that for many pupils and their teachers the possession of such a rule, and ability to use it, was what they meant by 'understanding'.*

First, we looked at a couple of recorded pupil 'answers' to some typical textbook 'questions'.

One kind of reaction to this 'question'…

Remove brackets from: $3(2a + 4c)$

… is this howler 3 2a + 4c (response A)

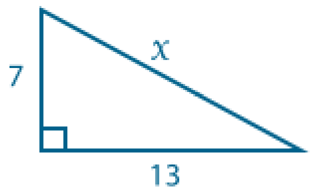
Another error perhaps reflecting the same misreading of the task is to answer…

Expand the brackets: $3(2a + 4c)$

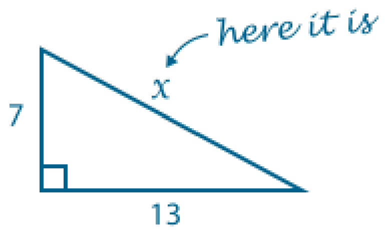
… by writing this… 3(2a + 4c) (response B)

Likewise, given this to do …

Find *x*



a pupil responded in this way...



(response C)

When we've stopped laughing – or perhaps groaning – over these pupils' misinterpretations of what they were expected to do, we can think about the nature of the errors. The instruction has been taken as requiring action on the presented surface form of the presentation, rather than upon a mathematical interpretation of it.

Are these learners' reactions mistakes? No, because, according to the pupils' interpretations, the questions have been successfully answered. But these superficial responses (A, B and C) do not indicate either 'instrumental' or 'relational' understanding.

In contrast, some pupils make errors that show a little 'instrumental' understanding, although inappropriate. For example, if a pupil who is asked in a textbook to…

Collect like terms : $2a + 3b + 5 + 3a + 2b$

…writes… $2a + 3b + 5 + 3a + 2b = 15ab$ (response D)

…a different kind of error is manifest. The pupil has done more than just look at the way numbers and letters are presented on the page. The pupil has interpreted the question as requiring some kind of mathematical procedure – because the sum of the numbers has been calculated.

Response D might have been inhibited by better 'relational' understanding, although the pupil may well have not responded at all – aware perhaps that the form $2a + 3b + 5 + 3a + 2b$ requires interpretation, but not what it should be.

Teachers commonly use images such as number lines and shapes to display arithmetical and algebraic properties, but there is less stress on pupils using reasoning to convince themselves of properties – which we may lament if we believe (see references below) that a lesson in which there is neither generalisation nor proof is not a mathematics lesson. If this sounds extreme, we may comfort ourselves by noting that mathematics as the practice of reason need not be part of every lesson labelled 'Mathematics'.

Relational understanding of the expression $3(2a + 4c)$ may involve associating it with a particular property of combinations of operations, the distributivity of x over +. When you decide that particular pupils will attach a 'name' to this property it is like so much in teaching, a judgement call. Rather than saying 'this is the distributive law', and expecting pupils to accept it, and apply it, as an unquestioned 'law', the property may become evident to them if they investigate rearrangements of groups of objects, and create their own examples. If anything at all has to be said, "…and this property can be called distributivity" gives a better impression that it is to do with how numbers combine rather than something handed down by 'the Almighty'.

Using their understanding of distributivity, pupils will see that:
3 lots of (2 times a number + 4 times another number) =
3 lots of (2 times the first number) + 3 lots of (4 times the other number),
and that this makes
6 times the first number + 12 times the other number.

They then need only to know the conventional way of representing this generalisation to *understand* that $3(2a + 4c) = 6a + 12c$.

Similarly, with their understanding of distributivity pupils will be better placed to make sense of the following:
$2a + 3b + 5 + 3a + 2b = 2a + 3a + 2b + 3b + 5$
$= (2 + 3)a + (2 + 3)b + 5$
$= 5a + 5b + 5$

Pupils may accordingly be better placed to see how the commands 'expand the brackets' and 'collect like terms' are connected with one another.

More than one PGCE student has said "that's a long-winded way of doing it", to which I reply "It's not a method, it's a proof". That graduates should say it at all is a sad reminder that even after being exposed to generalisation and proof in their degree courses, some PGCE students are so soon dragged into a mindset that I can only characterise as non-mathematical.

Pupils can be trained to present, or 'lay-out', their reasoning in a particular 'standard' way. Whilst this is not always inappropriate, it can be a pernicious attempt to address pupils' needs, and desires, for relational understanding – by substituting mere trained behaviour.

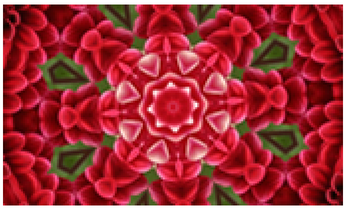
I don't mind slick nonsensical 'rules', like 'change the side, change the sign', or 'two minuses make a plus', provided they are mere triggers, or *aides memoires*, to procedures which they know have mathematical reasons.

The difficulty of engendering 'relational' understanding is not to be underestimated. Introducing seemingly arbitrary procedures tends to produce a smooth progression of a kind. 'Skill'-teaching may proceed with smooth uniformity, but relational learning goes in fits and starts, and at times, with excitement, by leaps and bounds. You can teach 'skills' by rote, but understanding by rote is a self-contradiction.

Further reading

- Thinkers by Chris Bills, Liz Bills, John Mason, Anne Watson. Pub. ATM; a collection of activities to provoke mathematical thinking.
- *Developing Thinking in Algebra* by Mason, Graham Johnston-Wilder. Pub. Sage; the course book for the OU CPD module of that name. There is a review on the ATM website.

Secondary Magazine
Issue 65



National Centre
for Excellence in the
Teaching of Mathematics

5 things to do this fortnight

- If you are planning lessons about symmetry, why not sit back one evening and listen to Professor Ian Stewart of the University of Warwick explaining in his short history of symmetry, Why Beauty is Truth.

- In gentle preparation for injecting new strategies and activities into your teaching next term, you could look at The Gaps and Misconceptions Tool of the Association of Teachers of Mathematics. The collection of Mental Images for Division, Fractions, Decimals, Percentages, Ratio and Proportion, and Subtraction contains lots of interesting and effective ideas to use with students, and with which to address difficulties.

- Have you explored Researchers in Residence? Having a young researcher in your classroom may motivate your students, opening their eyes to new possibilities, while helping you monitor students' progress in an even more exciting classroom atmosphere.

- During your break from the classroom you could put aside an hour or so to explore possible professional development opportunities for yourself and your colleagues. Very many and varied regional and nationwide courses and events are described in the Professional Development Directory, with links to providers.

- For relaxation and inspiration you might visit the beautiful, and very informative, virtual art and mathematics exhibition, Surfaces beyond the Third Dimension by Thomas Banchoff.

Secondary Magazine Issue 65

National Centre for Excellence in the Teaching of Mathematics

Diary of a subject leader

Issues in the life of an anonymous Subject Leader

The end of the GCSE tunnel is in sight. On Thursday is the final GCSE Statistics after-school revision session and it's good to see nearly 20 of the 62 students turn up. As always, we provide some food for them since the brain needs sugar to keep it going. The only kind of drink left is Irn-Bru, and the cans of that are drunk. As the session proceeds, the students become strangely ever more fired up. Later a colleague informs me that Irn-Bru contains a very high level of caffeine!

On Friday morning we do a final booster session from 8.30 to 9.00, looking at some common 'errers' – then off they go for a two-and-a-half-hour examination. The paper looks less 'friendly' this year. There's a question that throws them a bit – the students automatically fill in a table, and then, much later, the question says 'fill in the table'! Dilemma! Do I tell them there's nothing wrong with the question, leave it at that and have them worrying? Or do I explain that they have just done that part of the question a bit earlier than requested? I smile benignly and tell them not to worry.

I think I'm due to travel to Birmingham the following day, but am slightly worried that I haven't had any papers for the meeting, so I email to see where the papers are. I get the reply that they are not ready yet since the meeting is in fact two weeks later. So – a bonus weekend at home!

I attend meetings in school the following Monday (Pedagogy) and Tuesday (Governors Chairs' Review in the morning and Curriculum and Strategic Improvement (CSI) Committee, which I chair, in the evening). It's my final CSI meeting since I'm standing down after twelve years as a governor. It is very interesting being a governor; I've gained insight into how this team of people support the school and make sure that the future generation receives the best education possible with the available resources. I thoroughly recommend it! Find out more at Governornet.

Wednesday arrives and I travel up to Doncaster at the end of the school day. I've been invited to give the opening talk to the NANAMIC (National Association for Numeracy And Mathematics In Colleges) annual conference on Thursday morning. All goes well for an Able Seaman of HMS Pickle, the ship that brought the news of Trafalgar back to Britain in 1805.

In the afternoon I drive back south since it's the Y11 Prom on Thursday evening, and I do the photographs for that. This year my younger daughter helps organise the students who arrive by a variety of means of transport including synchronised space hoppers, tractors, fire engines and a Hell's Angels' escort. I take 530 photos, which my daughter drops off for 9x6 processing (I found out many years ago that a print size of 7x5 reduces the end of the shot – 6x4 and 9x6 both have a ratio of 1:5, but 7x5 uses 1:4 so a bit off the end of each negative is not printed).

I'm up early the following morning and back on the road, this time up to York for the Texas Instruments T^3 Annual Conference at the National STEM Centre. This was a fantastic conference with a wide variety of presentations in a great centre. Professor Celia Hoyles gives the opening plenary, the first of many interesting and motivating speeches about aspects of mathematics and science education. I recommend this great set of free resources from Texas Instruments.

At the evening meal in the Barley Hall the food is superb. I feast on corned beef pie, a delicacy I haven't had since leaving the North East in 1993. When I complement the caterers on their recipe, I find that they used to live very near to where we had our first house. We reminisce about local village names such as No Place and Pity Me.

When the conference finishes at lunch on the Sunday I'm whacked! I've chaired a one-and-a-half-hour discussion session, co-led a mathematical walk, and given two other 'presentation' sessions. But the commonly felt euphoria about doing mathematics doesn't make this feel like work. I drive home to find all my immediate family there – it's my wife's birthday. It's probably going to cost me a kitchen refit this year since I've missed most of it!

On Monday we have our faculty meeting after school in which we start by setting years 8 and 9 for next year. That doesn't take long, so we look at the faculty SEF, which I've been working on with my deputy, and then with my line manager. The meeting has to finish on time, because three of us are due back at 6pm for the Y6 Parents' evening. I dash home, boil a few eggs (which our admin assistant sells from her own hens – absolutely fantastic taste and often double-yolkers), get them eaten with plain bread and some salt and pepper, then straight back to school.

I'm there to sell calculators and stuffed pencil cases, to help the young students feel confidently prepared when they start in September. I've developed my own patter: 'Best value this side of the North Pole', 'Guaranteed to hold the secrets of mathematics – but you have to make them come out in the right order', 'Calculator? Left-handed or right-handed?' (I hand them over with my left or with my right hand). We had prepared 200 stuffed cases and ended up with only two left at the end of the evening – not bad estimating!