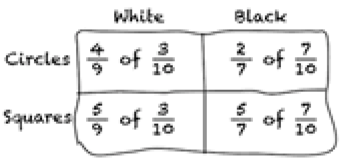
Secondary Magazine 144

National Centre for Excellence in the Teaching of Mathematics

Hello and welcome to the first Secondary Magazine of the 2017/18 school year. With new curricula aplenty, we take stock of the reasoning/problem-solving challenges of the first set of GCSE 9-1 exams, and suggest teaching approaches to make sure your students can meet them. And with the first teaching of the new A level courses, our second article helps signpost information, training and resources that you might have missed so far. Don't forget that all previous issues are available in the Archive.

This issue's featured articles



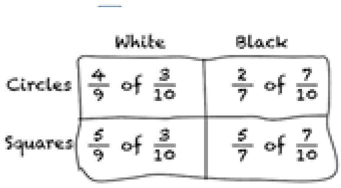
Addressing the Reasoning and Problem-Solving demands of the new GCSE
On top of the new GCSE content this summer, it was the increased emphasis on reasoning and problem-solving that risked keeping teachers up all night wondering whether their students had the right approaches and necessary skills. This article looks in detail at how teachers might best prepare students for the sort of challenges presented by the new exam papers.

Catching up with the New A Level - a whistle-stop tour of changes, resources, CPD and more
Having hardly drawn breath after results from the new GCSE, teaching for the new A level has begun… How is it going? Are you feeling confident, competent and sufficiently well-prepared to teach the new content? If you weren't as well-prepared in advance, as you might like to have been, here we signpost information, resources and training to help you get up to speed.

And here are some other things for your attention:

- There's a wealth of material of interest to secondary maths teachers in the latest newsletter from our consortium partners, Mathematics in Education and Industry (MEI). Of note this month is their free problem-solving guide, designed to address practically, the increased problem-solving emphasis at GCSE and A level; Factris, a new game app from MEI that quickly impresses the importance of factors using falling, re-sizeable rectangles to build a wall; and plenty of analysis of and response to the Smith Review into post-16 Mathematics, published in July
- In his latest blog, the NCETM's Director, Charlie Stripp, reflects on the emerging landscape of post-16 Maths following the Smith Review, particularly addressing concerns about participation rates. He argues that students with GCSE grade 5 should be able to tackle A level Maths. This was the subject of a feisty debate in last week's #mathscpdchat discussion - you can join in every Tuesday 7-8pm on Twitter
- A wide-ranging programme of school-led teacher professional development has been put together by the Maths Hub network for 2017/18, building on the most successful and effective parts of last year's programme. You can find out what's on offer here, and contact your local Maths Hub for further details.
- A reminder about our Qualifications and Curriculum (Q and C) page, where we update any policy or curriculum changes of an official sort of nature. Particularly aimed at Secondary Heads of Department, the page can be filtered by phase, exam or topic. You can sign up for updates on Twitter (@NCETMQandC), or through the Q and C updates community.
- STEM Learning has a rich offering of maths CPD this year, much of it funded by Enthuse bursaries, hugely reducing costs to schools. Their website has full details of what's available.

Image credit: Page header by Alex Wing (adapted), in the public domain



New, practical, interactive resources, from MEI and IET – FREE

Addressing the Reasoning and Problem-Solving demands of the new GCSE

In 2017/18, a Maths Hubs National Collaborative Project (NCP17-12), will be developing teaching approaches to match the new challenges of the new GCSE (9-1). They will be particularly addressing 'challenging topics' – those that students did less well with in the first exams this summer. The Secondary Magazine will be following and publicising some of their work in order to support GCSE teachers in the classroom. We begin in this edition, by considering how you might address the increased emphasis on Reasoning and Problem Solving.

This is an abridged version of a longer article *which can be downloaded as a PDF. "This article contains activities for the classroom to support students in the development of reasoning and problem solving skills. These are at the foot of the article.*

Many of the 2017 GCSE Mathematics questions require pupils to be willing '*to face the unexpected and to think how to link known techniques into effective solution chains*'. (Anthony D. Gardiner, Teaching Mathematics at Secondary Level, 2014).

Some questions are particularly difficult because pupils must find a starting-point for themselves. Also the construction of an adequate response depends heavily on the pupil's ability to reason mathematically, to argue 'if … then …', to provide justifications for facts that they state. In the 2017 GCSE papers, questions of this kind can be sorted into three broad categories:

1. 'Work out …' problems where something particular has to be found
2. algebraic proofs about number
3. geometric proofs.

When selecting or designing tasks to help pupils develop the skills needed to succeed with these questions it is helpful to be guided by some general principles that apply to all three categories:

- Challenge pupils to find as many mathematical relationships as possible between the constituent parts of situations about which it is possible to reason mathematically (see Task 1 below)
- Provide opportunities for pupils to see things for themselves. Then challenge them to try to convince other people that what they see *must be true because* … .
- Given a problem, challenge pupils to construct their own 'similar' problems by varying some aspect, or aspects, of it.
- Design tasks that will help pupils develop the habit of laying out calculations and deductions line-by-line so that a sequence of successive steps can be seen as a single chain of reasoning.
- Challenge pupils to think of different chains of reasoning to a particular result.
- Design tasks that will help pupils develop the habit of simplifying calculations and expressions wherever possible.

Before they reach Key Stage 4 pupils should be learning to use reasoning to solve multi-step word problems where they have to 'work out' or 'find' something. These reasoning skills develop into skills required in the construction of proofs. So first we consider preparation for, and approaches to, problem-solving of the first kind above - 'work out' problems, where something particular has to be found. (For detailed treatment of algebraic and geometric proofs, please refer to the full article).

Part A: 'Work out …' problems where something particular has to be found

Example 1: 2017, Edexcel, Foundation Level, Paper 3, Question 13

The size of the largest angle in a triangle is 4 times the size of the smallest angle.
The other angle is 27° less than the largest angle.

Work out, in degrees, the size of each angle in the triangle.
You must show your working.

Example 2: 2017, Edexcel, Higher Level, Paper 1, Question 14

White shapes and black shapes are used in a game.
Some of the shapes are circles.
All the other shapes are squares.

The ratio of the number of white shapes to the number of black shapes is 3 : 7

The ratio of the number of white circles to the number of white squares is 4 : 5

The ratio of the number of black circles to the number of black squares is 2 : 5

Work out what fraction of all the shapes are circles.

Example 3: 2017, OCR, Foundation Level, Paper 3, Question 19

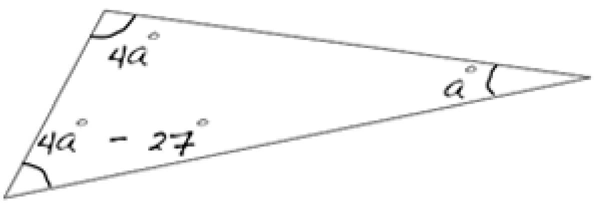
Two numbers have these properties.

- Both numbers are greater than 6.
- Their highest common factor (HCF) is 6.
- Their lowest common multiple (LCM) is 60.

Find the two numbers.

These questions are hard because pupils have to decide for themselves where to begin. A powerful first strategy is to represent to oneself all the information that you can extract (that easily follows) from the given information. Ask pupils: 'What does the given information tell you? What do you know? How can you show, perhaps in a sketched diagram or chart, what you know?'

In ***Example 1*** the given information could be represented in a sketch of a triangle:



In ***Example 2*** information given in ratio form can be converted to information in fraction form: 3/10 of all the shapes are white, 7/10 are black, and so on. These fraction facts could be represented in a diagram:

	White	Black
Circles	$\frac{4}{9}$ of $\frac{3}{10}$	$\frac{2}{7}$ of $\frac{7}{10}$
Squares	$\frac{5}{9}$ of $\frac{3}{10}$	$\frac{5}{7}$ of $\frac{7}{10}$

In ***Example 3*** initial possibilities could be listed:

- the HCF is 6 means that both numbers are multiples of 6:
  (6), 12, 18, 24, 30, 36, 42, 48, 54, 60, …
- the LCM is 60 means that both numbers are factors of 60:
  (1, 2, 3, 4, 5, 6), 10, 12, 15, 20, 30, 60.

The process, exemplified above, of representing-to-themselves all the information that they can glean from word statements is a process that pupils can practise. Challenge pupils to represent-to-themselves all that they can derive from some information (an ‘information-list’), for example as in:

Task 1 *(see Task 1 at the foot of this article)*

When pupils have represented-to-themselves (and shared and discussed) what any list of statements about a situation tells them, ask them to suggest what they might be asked to ‘work out’ or ‘find’. Collect and display their suggestions so that they can think about each-others’ ‘questions’. It may be that, in showing on paper what they gleaned from the stated facts, some pupils have already reasoned to the answers of some of the suggested ‘questions’. However, if they have not yet connected what they now know (have jotted down) with what they *want* (the answer to a suggested ‘question’), they will have to *bring in* to the collection of facts they are reasoning about *other facts* from their repertoire of known properties and general relationships. This process, of *deciding what mathematical-knowledge-not-provided-with-the-given-information will reveal more facts*, can also be practised, and is exemplified in:

Task 2 *(see Task 2 at the foot of this article)*

In addition to practising …

- **representing** in a precise structural way, facts and relationships conveyed **at first** to you via word statements
- **identifying** mathematical knowledge and procedures that can be applied to a situation **to extend** what you know about it

… pupils should practise …

- **writing** complete **chains of reasoning**.

Writing complete chains of reasoning helps pupils to identify precisely where ‘if … then …’, ‘because …’ and ‘therefore …’ occur in their thinking, and so learn to apply similar thinking in new and different situations.

Pupils need to learn to lay out calculations line-by-line, with:

- given information and any symbols representing ‘unknowns’ declared
- each fresh step on a new line (and any explanation given alongside)
- the final answer clearly displayed at the end.

‘The sequence of successive steps can then be grasped as a single chain of reasoning in which each step follows clearly from those that went before.’ (Anthony D. Gardiner, *Teaching Mathematics at Secondary Level, 2014)*

Example chains of reasoning for a problem created from information in Task 1

If, when solving ‘Work out … problems’, pupils *have had plenty of practice in thinking their own ways through to solutions* and *then communicating* their thinking by writing-out *complete chains of reasoning*, the step to *constructing proofs* is not so great.

There is another aspect of working mathematically with which pupils need to develop competence *in order to cope successfully with all sorts of ‘Work out …’ problems*. It is the routine simplification of numerical expressions in a way that exploits structure (‘structural arithmetic’). (Ref: Anthony D. Gardiner, as above). When evaluating numerical expressions it is almost always quicker to simplify wherever possible than to ‘calculate blindly’. For example, in solving *Example 2 of the 2017 GCSE questions shown above*, a numerical expression occurs that can be evaluated rapidly like this:

$$\frac{{}^{2}\cancel{4}}{{}_{3}\cancel{9}} \times \frac{\cancel{3}}{\cancel{10}_{5}} + \frac{{}^{1}\cancel{2}}{\cancel{7}} \times \frac{\cancel{7}}{\cancel{10}_{5}} = \frac{2}{15} + \frac{1}{5} = \frac{2}{15} + \frac{3}{15} = \frac{5}{15} = \frac{1}{3}$$

Pupils need to learn through lots of practice how numerical expressions can be simplified.

Using structure to simplify numerical expressions: examples

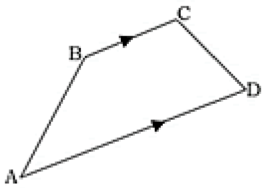
When pupils are habitually looking for ways to use structure to simplify numerical expressions they internalise meanings, structures and procedures that they can then draw on when working with algebraic expressions while trying to *construct proofs*.

The full article can be downloaded as a PDF. What follows are the tasks suggested in the article

Task 1

This task is about extracting and representing as much information as possible. It is not, at this stage, about finding answers to any problems. Emphasise to pupils that the task is to represent, as concisely as possible without merely writing more word sentences, what the given word statements tell them. Encourage the use of sketched diagrams, charts, numbers and symbols.

Example information-lists

A	B	C
 ABCD is a trapezium with BC parallel to AD. BC = CD AB = DB Angle BCD is 60° greater than angle BAD.	ABCD is a square drawn on a coordinate grid. The centre of the square is at the origin. A is the point (2, 1). The gradient of the diagonal CA is $\frac{1}{2}$. The y-coordinate of B is negative.	This information is about a rectangle and two squares. The side-lengths of the rectangle are in the ratio 2 : 5 The area of square A is 90% of the area of the rectangle. The side-length of square A is $\frac{3}{4}$ of the side-length of square B.

Task 2

Give pupils some information-lists (such as A, B or C in Task 1, above), and challenge them with:

> *'What mathematical facts could be added, and what mathematical processes could be applied, to each situation in order to reveal more information?'*

Pupils could compare and discuss their suggestions.

By working on tasks as outlined above pupils could arrive at the following kind of analysis (shown for information-lists A, B, C, in Task 1):

List A

<table>
<tr><th>Adding this mathematical knowledge / applying these mathematical procedures ...</th><th>... transforms this representation ...</th><th>... to this representation.</th></tr>
<tr><td>

- base angles of isosceles triangle are equal
- alternate angles (parallel lines) are equal
- angle sum of triangle is 180°

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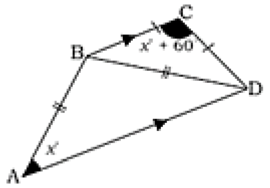

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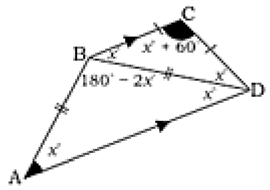

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- form and solve an equation:

$$3x + 60 = 180$$
$$3x = 120$$
$$x = 40$$

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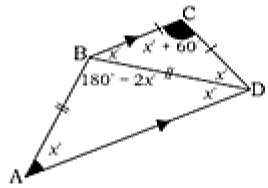

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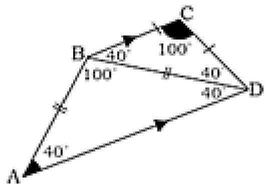

</td></tr>
</table>

List B

<table>
<tr><th>Adding this mathematical knowledge / applying these mathematical procedures ...</th><th>... transforms this representation ...</th><th>... to this representation.</th></tr>
<tr><td>

- the centre of a square is at the mid-point of its diagonals

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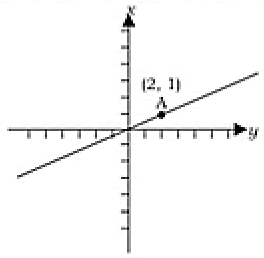

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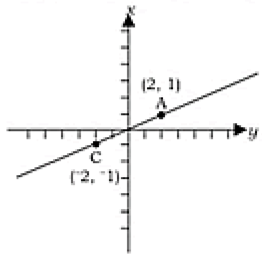

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<tr><td>

- the diagonals of a square bisect each-other at right angles

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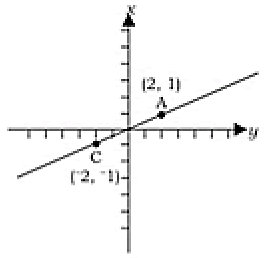

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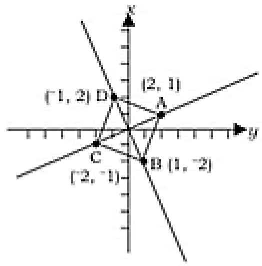

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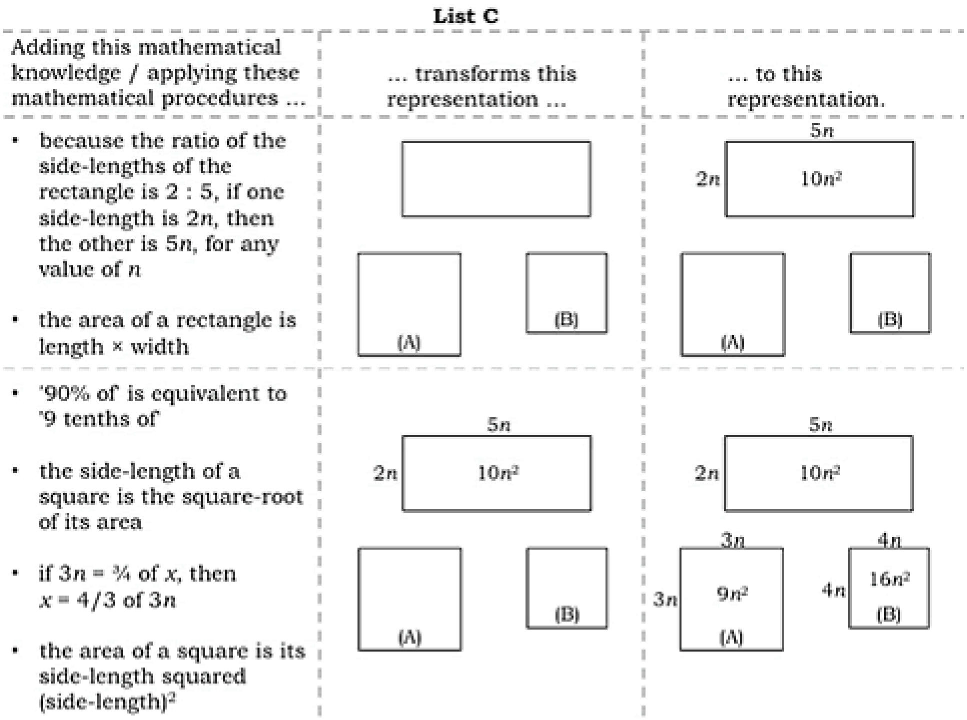

List C

Adding this mathematical knowledge / applying these mathematical procedures ...	... transforms this representation ...	... to this representation.
• because the ratio of the side-lengths of the rectangle is 2 : 5, if one side-length is $2n$, then the other is $5n$, for any value of n • the area of a rectangle is length × width	(A) (B)	$5n$, $2n$, $10n^2$, (A) (B)
• '90% of' is equivalent to '9 tenths of' • the side-length of a square is the square-root of its area • if $3n$ = ¾ of x, then x = 4/3 of $3n$ • the area of a square is its side-length squared (side-length)^2	$5n$, $2n$, $10n^2$, (A) (B)	$5n$, $2n$, $10n^2$, $3n$, $3n$, $9n^2$ (A), $4n$, $4n$, $16n^2$ (B)

Example chains of reasoning for a problem created from information in Task 1, List A

- Suppose List A in Task 1 is turned into a 'Work out … problem' by adding 'Work out the size of angle ABD'. At least two different complete chains of reasoning to the solution of this problem could be constructed, possibly as follows:

Let angle BAD be x°

Then angle BCD = $x^\circ + 60^\circ$ (given)

Angle BDA = x° (equal base angles of isosceles triangle ABD)

Angle CBD = x° (alternate angles, BC parallel to AD)

Angle CDB = x° (equal base angles of isosceles triangle BCD)

Therefore $(x + x + x + 60)^\circ = 180^\circ$ (angle sum of triangle BCD)

Therefore $3x + 60 = 180$

Therefore $3x = 120$

Therefore $x = 40$

Therefore angle BAD + angle BDA = 80°

Therefore angle ABD = 100° (angle sum of triangle ABD)

Or

Let angle BCD be x°

Then angle BAD = $x^\circ - 60^\circ$ (given)

Angle BDA = $x^\circ - 60^\circ$ (equal base angles of isosceles triangle BAD)

Angle CBD = $x^\circ - 60^\circ$ (alternate angles, BC parallel to AD)

Angle CDB = $x^\circ - 60^\circ$ (equal base angles of isosceles triangle BCD)

Therefore Triangle ABD is similar to triangle BCD (corresponding angles are equal)

Therefore $2(x - 60)^\circ + x^\circ = 180^\circ$ (angle sum of triangle BCD)

Therefore $3x - 120 = 180$

Therefore $3x = 300$

Therefore $x = 100$

Therefore angle BCD = 100°

Therefore angle ABD = 100° (angle ABD and angle BCD are corresponding angles in similar triangles)

Using structure to simplify numerical expressions: examples

$82 + 57 + 18 + 43 = 100 + 100 = 200$

$16 \times 75 = 4 \times (4 \times 25) \times 3 = 1200$

$29 \times 53 + 29 \times 47 = 29 \times (53 + 47) = 29 \times 100$

$\frac{4.2}{6.3} = \frac{42}{63} = \frac{6 \times 7}{9 \times 7} = \frac{6}{9} = \frac{2}{3}$

$\sqrt{(4 + 2\sqrt{3})} = \sqrt{(1 + 2\sqrt{3} + 3)} = \sqrt{((1 + \sqrt{3})^2)} = 1 + \sqrt{3}$

Catching up with the new A level - a whistle-stop tour of changes, resources, CPD and more

Having hardly drawn breath after results from the new GCSE, teaching for the new A level has begun… How is it going? Are you feeling confident, competent and sufficiently well-prepared to teach the new content? Where do you feel you still need more support? Have student numbers held up? Does the new GCSE seem to have produced students more able to meet the demands of A level? If you weren't as well prepared in advance, as you might like to have been, here we signpost information, resources and training to help you get up to speed.

Importantly, 100% of the curriculum content for A level maths is now prescribed by the DfE meaning that choice of exam board will be made by other factors, and can possibly be made later. This also means that teachers should not feel tied to 'the' exam board textbook but can hunt for the one they feel is most appropriate to the way they want to teach the course, or mix and match. As you'll probably already know, the course is now linear, with all assessment in the final exams (three two-hour papers), and all students will study Pure Maths, Mechanics and Statistics (boards differ in how they distribute assessing these three areas over the three papers).

For Further Maths, 50% of the content is prescribed – there is more flexibility in the other 50%. It is assessed at the end of the course in six hours of exams (boards vary in numbers of papers), containing some Pure and some Applied Maths.

If you'd like a quick tour of all the changes, Mathematics in Education and Industry (MEI) provides a useful summary. Answers to any further questions you have, may well be answered in this list of FAQs. A detailed description of *content* changes is available in this table from the Further Maths Support Programme (FMSP). MEI also explains the increased emphasis on embedding technology into the course.

If you are taking this opportunity to re-write your A level Scheme of Work, you might like to use the editable scheme of work that MEI has put together. This is not board-specific. For Further Maths, FMSP provides these editable schemes of work for each board.

And to keep you updated in all matters 'Qualification and Curriculum', visit the NCETM Qualifications and Curriculum page and subscribe to updates through our online community or Twitter feed (@NCETMQandC).

If you are hunting down resources, specifically for the new content, Jo Morgan at Resourceaholic is ahead of the game tracking down free online material for the new specification, while her older pages cover topics that remain from before. STEM Learning's A level resource packages also do an excellent job of pulling together the best of the resources to save you having to do the hunting. Underground Mathematics provides excellent, creative and connected free learning resources, now hosted by NRICH, with CPD on Underground Mathematics provided by MEI. If you have some budget available, then MEI's Integral resource provides high quality, comprehensive resources for use in the classroom and for students' independent study (though much of this – particularly the further maths and applied content - is available free by registering with FMSP). FMSP's A level Problem Solving Resources give you a chance to focus on the increased problem-solving emphasis. Exam boards also all offer their own selection of resources.

In terms of teacher professional development with a particular focus on the new A level, the exam boards all offer types of short training. From Edexcel's free *Getting Ready to Teach* webinars and local collaborative networks, to OCR's free face-to-face sessions or webinars, tailored to particular parts of the curriculum under the headings 'Get Started', 'Get Going' and 'Get More', and AQA's mix of online and face-to-face courses focusing mainly on the applied modules and the requirement to embed technology in the curriculum. MEI ran a lot of their teacher CPD over the summer, but still offer bespoke courses from their experts, that schools can buy in for training days. STEM Learning's CPD is more extensive and in-depth: residential courses of three or four days, attracting full bursaries so that teachers attend for free. They are Teaching for deep understanding in A level mathematics, and the summer school New to teaching A level mathematics.

The Maths Hub network is this year running a national project under the title 'Supporting the Teaching of the New A Level Mathematics Using Technology' and each Maths Hub will be recruiting schools and teachers to work in depth on this subject, in local Work Groups. Contact your local Maths Hub if you are interested in taking part.

Sometimes it's just nice to know how others are getting on… this blog follows the developments in one maths department.