Secondary Magazine 114

National Centre for Excellence in the Teaching of Mathematics

Happy New Year…well, you know what we mean. Ever thought about teaching in China? Term started there on 1 September, and the first national holiday was on 8 September – followed by Teachers' Day, when all pupils have to give cards and presents to their teachers!

The team at the NCETM hope that you have had a refreshing summer break, and have come back to school re-invigorated for the mathematics journey ahead. We have decided to update the format of the Secondary magazine, and introduce some new items to stimulate, inspire and enhance your teaching. Let us know what you think.

Contents

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Heads-Up

Whilst not wanting to define this article as mathematical gossip, it does come close! We've brought together news and current mathematical affairs, all in one place. We do hope it will interest you.

You may have heard in the news about the recent Fields Medal awards, the so-called "Nobel Prizes" of Maths. The media gave lots of attention to the awards as this is the first time that a Fields Medal has been awarded to a woman. Somewhat surprisingly, the president of Iran tweeted his congratulations to Dr Mirzakhani, who now works in the USA, and he "liked" a photograph of her in which she was not wearing a headscarf in a public place.

As you are reading this, you have no doubt made a detailed analysis of the A level and GCSE results for your pupils. At A level, despite a decline in overall entries, Mathematics increased its entry numbers and overtook English to become the most popular A level subject in 2014 with 88 816 entries. Although the JCQ figures show that both Mathematics and Further Mathematics continue to have many more entries from boys, the gap between the proportion of boys who gain an A or A* grade and the proportion of girls is narrowing (4% more boys doing so in 2013 has dropped to 3.1% more boys in 2014).

GCSE results for Mathematics also improved in 2014, with the national A*-C percentage rising from 57.6% in 2013 to 62.4%. There were fewer entries (736 403 in 2014 compared to 760 170 in 2013, mainly because there were far fewer early entries, a change of school policy that the NCETM endorses) but the percentage of A*/A grades increased from 14.3% to 15.2%. Did your examination results follow these trends?

In response to popular demand, the NCETM has uploaded the first draft of a series of progression maps for the new KS3 programmes of study. This resource suggests how to divide the content of Algebra, Number, Ratio and Proportion and Rates of Change, Geometry and Measures, and Probability and Statistics across Y7-9. They are very much a work in progress, and we look forward to hearing from you how they can be enhanced and developed further.

There is a lot of talk about "maths mastery" at the moment. To prompt your thinking about what this means in your classroom, read the latest thoughts from the NCETM Director Charlie Stripp on Maths, Mindsets and Mastery in his blog, Charlie's Angles.

As the new term starts, you may want to encourage your students to consider taking mathematics in the sixth form; they may be asking you for advice. A recent BBC news story revealed that students often did not feel that they had appropriate A level advice to enable them to take the next step in their chosen careers. Informed Choices, a Russell Group publication, gives good advice to students, in particular about the importance of studying what are called facilitating subjects, of which two are Mathematics and Further Mathematics. It's well worth your reading, so that you know you are maximising your students' chances of future academic success.

Looking ahead, the A level Content Advisory Board (ALCAB) has published reports linked to its deliberations on the future content of A levels, with links to the related DfE and Ofqual consultations: why not make time to record your comments?

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Building Bridges
One third...of what? The interpretation of fractions in KS2 and KS3

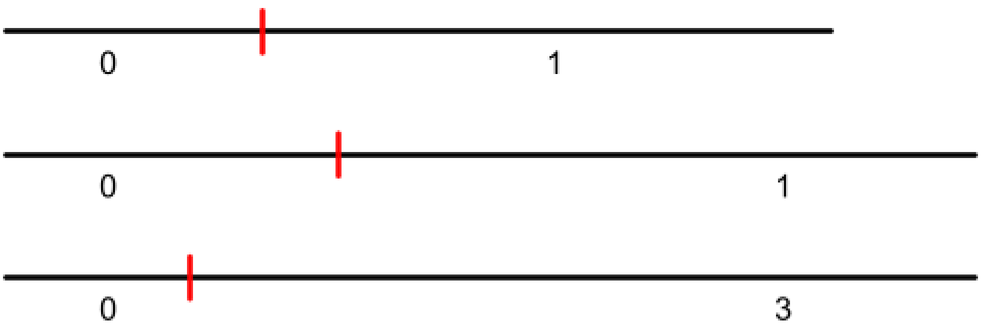
KS3 pupils often find fractions difficult. Some progress further than others in KS2, so that in your class there could be some who can confidently and accurately add and subtract fractions, some who will have already seen the routines for doing so but don't remember how to apply them successfully, and some who haven't yet met these routines. This will change in the future because the operations of arithmetic with fractions (add, subtract, multiply and divide them) is, from now, part of the KS2 programme of study, but nonetheless classes will still span the spectrum of confidence, accuracy and conceptual understanding. The new programmes of study emphasise that teaching must develop

- fluency
- reasoning
- problem solving

and so it's not sufficient that pupils only memorise algorithms: a deeper conceptual understanding of fractions must, over time, be embedded.

Pupils find fractions difficult to understand for a good reason: they are! The great minds of 19th century mathematics took some time, great pains and much ink to develop formally and securely the concept of the rational number. At school level, the idea of a fraction is hard because pupils encounter it in one context – as a proportion, a part of a whole – and then, later, start to use it in arithmetic as a number: they first see the fraction $\frac{1}{3}$represented as 1 slice of a cake which was first cut into 3 slices, or 1 piece of a chocolate bar which was first broken into 3 pieces, and then later they see $\frac{1}{3}$ as part of a sum such as $\frac{1}{3} + \frac{3}{5}$, apparently in the same way as they calculate 4 + 17. A "part of something" has become a "stand-alone number": that is a conceptually challenging shift, and pupils are right to be uncertain about it. Of course, the same is true of 4, which is first met by pupils as the concrete result of a count ("4 apples") and then becomes an abstract number, 4.

Before revising or teaching the arithmetic of fractions, it's important to address this apparent change of status. Honesty is the best policy: pupils should be told that the written symbol – the mark made with ink on paper – that looks like "$\frac{1}{3}$" represents more than one thing: it can be used to specify the process "take one third of" (as in $\frac{1}{3}x$) or it can locate a place on a number line, that place being one third of the way between 0 and 1 if you start at 0 (just as the ink on paper mark "2" can be thought of as locating the place that is twice as far from 0 as from 1 if you start at 0). It can be helpful to show this on a selection of number lines: on each of the following, the red line is located at the number represented by writing "$\frac{1}{3}$".



The connection between the familiar "take one third of" process and the less familiar "a place on the number line" is made explicit by these diagrams: the standalone number $\frac{1}{3}$ is placed on the number line one third of the way between 0 and 1, starting at 0. There is a good opportunity here to link division with its inverse operation multiplication. The number line model shows that three pieces of length $\frac{1}{3}$ will fit (and fill) a gap of length 1, i.e. $3 \times \frac{1}{3} = 1$. Necessarily, therefore, $1 \div 3 = \frac{1}{3}$, and $1 \div \frac{1}{3} = 3$.

Calculation such as $2 \div 5 = \frac{2}{5}$ can be similarly represented: the place which is located two fifths of the way between 0 and 1 is the place that marks the end of one of the five pieces you get when you divide a length 2 into 5 equal parts.

Since $2\frac{1}{2}$ pieces of length $\frac{2}{5}$ will fill the gap of length 1, this means that $2\frac{1}{2} \times \frac{2}{5} = 1$. Therefore $1 \div 2\frac{1}{2} = \frac{2}{5}$, and $1 \div \frac{2}{5} = 21/2$.

Recently, UK teachers observing lessons in Shanghai schools have noticed that every opportunity is taken to emphasise the inverse relationships between the operations + and – and × and ÷. Whenever a sum such as $7 + 13 = 20$ arises, the equivalents $20 - 13 = 7$ and $20 - 7 = 13$ are elicited from the pupils; similarly $4 \times 3 = 12$ immediately prompts $12 \div 4 = 3$ and $12 \div 3 = 4$. This happens from grades 1 and 2 (equivalent to Y2 and Y3), and by grade 5 the pupils have abstracted this into a rule they can state and use: we would translate the characters they write as "one factor = product ÷ other factor".

As pupils develop this conceptual understanding of $\frac{1}{3}$ or $\frac{2}{5}$ representing a stand-alone number and not only specifying the process "take a part of" – as they realise that one third doesn't have to be one third "of" anything, then the idea of using it in arithmetic becomes much less mysterious. Next month we will consider ways to embed deep understanding of the algorithms of the arithmetic of fractions.

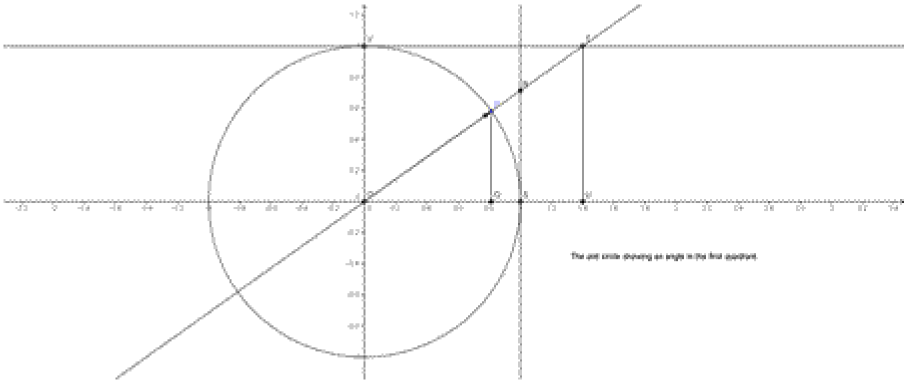
Image credit

Sixth Sense
Key representations in Level 3 Mathematics – The Unit Circle

The development and redefinition of trigonometric functions as functions of real numbers from the definitions of trigonometric ratios as scale factors between the side lengths of right-angled triangles is one of the 'big ideas' in Level 3 Mathematics.

The unit circle is a key visual representation in understanding this conceptual development. It allows the extension of the definition of all six trigonometric functions firstly to any angle in degrees, then to any angle in radians and then as functions of any real number, as well as providing a clear visual representation of all six trigonometric functions as displacements, and triangles which help establish the Pythagorean identities, some key trigonometric inequalities and the small angle approximations prior to differentiation of the trigonometric functions.

It is powerful when used in conjunction with trigonometric graphs (another key visual representation in trigonometry), and indeed, can be used dynamically to generate the trigonometric graphs themselves.



In the diagram above, if angle SOP = θ (measured anticlockwise from Ox, at the moment either in degrees or radians) then:-

- P is the point with coordinates (cos θ, sin θ) i.e. the displacement (i.e. the directed or signed distance) OQ represents cos θ and the displacement QP represents sin θ.
- R is the point with coordinates (1, tan θ) i.e. SR represents tan θ as a displacement along the tangent line to the unit circle at (1, 0).
- T is the point with coordinates (cot θ, 1) i.e. VT represents cot θ as a displacement along the tangent line to the unit circle at (0, 1).
- Additionally, OR represents sec θ as a displacement along the extended diameter OP and OT represents cosec θ as a displacement along the same extended diameter OP. "FYI": a line that cuts a circle at two points is called a secant lines; hence sec θ and cosec θ.

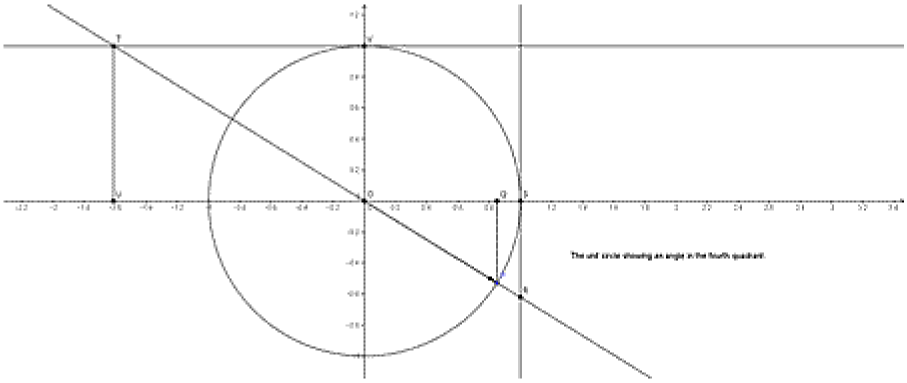
Note also that the triangles ΔOQP, ΔOSR and ΔOUT are similar, where it can be seen that

- sec θ/(1)= 1/cos θ, cosec θ/(1) = 1/sin θ and sin θ/cos θ = tan θ/(1) = 1/cot θ.

Using Pythagoras' Theorem in each of these same triangles gives

- $(\cos\theta)^2 + (\sin\theta)^2 = 1$, $1 + (\tan\theta)^2 = (\sec\theta)^2$, and $(\cot\theta)^2 + 1 = (\mathrm{cosec}\theta)^2$.

It is also worth noting that if angle SOP = θ (now measured in radians) and arc length SP = s, then θ = s and P is both (cos θ, sin θ) and (cos s, sin s) i.e. we can regard both θ and s as real numbers and we have defined the trigonometric (or circular) functions of any real number (as both θ and s can take any real value), rather than being limited to functions of angles.



The unit circle representation works perfectly for angles of any size and can be used to verify the results and statements above for angles in any of the four quadrants. It's well worth your students trying this for themselves for an angle in the fourth quadrant as shown in the diagram above. Even better, they can develop the representation in Geogebra (or Autograph) to revisit and use dynamically in class this term.

Finally, QP < arc length SP < SR so sin θ < θ < tan θ. Equivalently, dividing through by sin θ (which is positive for acute θ), we derive

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

This implies that, for small θ, sin θ is approximately equal to θ, because the centre fraction in the inequality is "sandwiched" between two expressions that tend to 1 as θ tends to 0. This also implies that tan θ is also approximately equal to θ (since tan θ = sin θ ÷ cos θ). These arguments can be made rigorous (and will be in higher study), but they are intuitively plausible and are accessible with the aid of the unit circle diagram. A similar argument starts with the observation that the area of the triangle OPQ is less than the area of the sector OPS, and both are less than the area of the triangle ORS: asking your students to develop this argument will be instructive and interesting for them.

The unit circle representation is a recurring representation throughout Level 3 Mathematics and can be easily scaled to any circle of radius r. Examples of its use can also be found in topics in Further Pure Mathematics such as polar coordinates and complex numbers, and in circular motion in Mechanics. Wherever it is a natural representation, trigonometric functions are likely to be helpful in formalising the mathematical description of the ideas or being explored. For example, considering the complex numbers a + bj and -b + aj as points on a circle of radius $\sqrt{a^2 + b^2}$ helps students see the geometrical interpretation of the effect of multiplying of complex number by j: an interesting result, and also one that contributes towards the motivation for the introduction of j into the number system.

Further reading:

Key Ideas in Teaching Mathematics: Research Based Guidance for Ages 9 – 19 (Anne Watson, Keith Jones and Dave Pratt) OUP 2013 Chapter 9: Moving To Mathematics Beyond Age 16

Image credit

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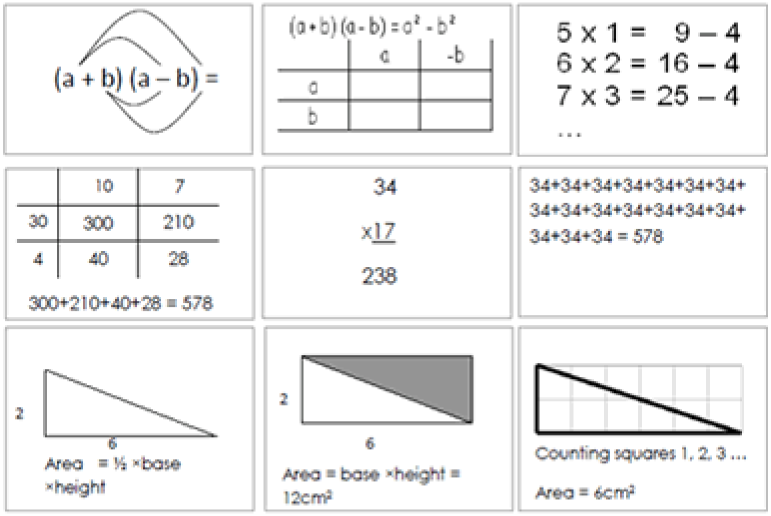
From the Library
Shh! No Talking!

In his article Relational Understanding and Instrumental Understanding (first published in Mathematics Teaching 77, 1976), Richard R Skemp defines two different types of understanding: relational and instrumental. You can read this extract (Handout 2 Page 4), and a summary of the article. Three of the key aims of the new National Curriculum Programme of Study are the development of conceptual understanding, of mathematical reasoning and of problem-solving skills; these all link to the relational understanding that Skemp defines.

You can explore practically relational and instrumental understanding with the card sort resource. The suggestion is to sort the cards into topic groups, and then form a continuum within each group from the cards that demonstrate a relational understanding to the cards that demonstrate an instrumental understanding. You – or, even better, your students – could then perhaps make up similar groups of cards of your own, or create an "understanding spectrum" (from infra-relational to ultra-instrumental!) for the classroom wall.



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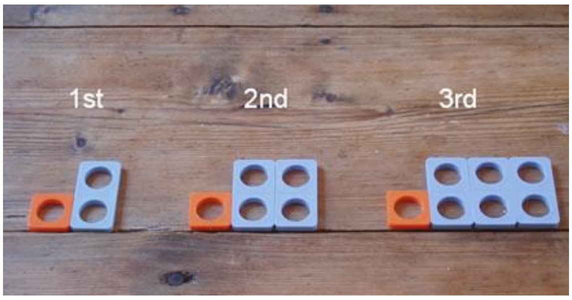
It Stands to Reason

For many students, one of their early encounters with using letters or symbols to represent numbers in secondary school is when finding an expression for the value of the nth term of a linear sequence. Although students will have had experience of algebraic representation in their primary schools, the associated task will, most likely, have been to find the value of the unknown symbol; the use of a symbol to mark a potential, or future, evaluation will be much less familiar.

The first part of the challenge of teaching this topic is to find out what they already know, and then build on that understanding to develop the reasoning skills that are the hallmark of good mathematicians. The NCETM has developed activities and resources to help. In the NCETM Departmental Workshop on sequences there is a PowerPoint presentation that encourages students to develop their understanding of the structure of a sequence and relate that structure to the growth of the sequence:



Seeing this, students may say

"In the first picture there is 1 orange piece and 1 blue piece"
"In the second picture there is 1 orange piece and 2 blue pieces"
"In the third picture there is 1 orange piece and 3 blue pieces"

Asking "what's the same and what's different" about the blue and orange pieces should elicit realisation that the blues are "double". At this point they may want to count the holes not the pieces; if counting a hole – a void – is found to be hard, you can ask "how many grains of rice would I need if I wanted to put one in each hole?"

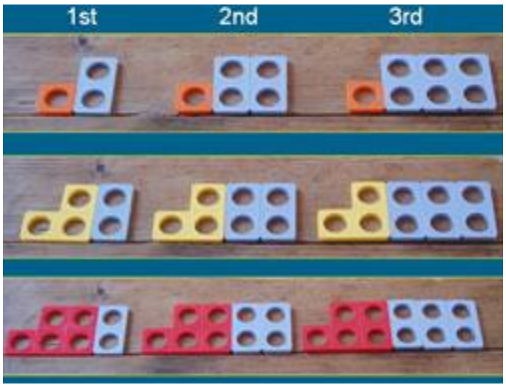
Then your students can be encouraged to rephrase their original answers in terms of the numbers of "orange holes" and "blue holes", and then predict that

"In the fourth picture there will be 1 orange hole and … blue holes"
"In the tenth picture there will be …"
"In the hundredth picture there will be …"

They then can be asked "How could you explain how many coloured holes there'll be in any pattern in the sequence?"

The use of colour is helpful because students can talk about the different parts of the sequence: the constant and the changing parts.

Then:



This slide allows students to make comparisons between three different sequences that all grow in the same way (or, and better, "at the same rate") but have different constant terms. Pupils develop their understanding of the structure of sequences, and use the observed structure to predict future specific values. The guiding question *"How could you explain how many holes there'll be in any pattern in the sequence?"* provides a link to talking about an expression for the value of the general term of the sequences of the number of holes in each pattern.

A natural follow-on from this resource is to give students an opportunity to create their own linear sequences using a practical resource that enables the use of two or more colours. The blog entry nth term from matchstick patterns gives some more suggestions for relating the structure of a sequence to its growth, and observes that *"John Mason suggests providing just e.g. the third diagram in a growing sequence. Following class agreement on the 'structure' of the growth, a generalisation can then be sought. The shared development, seeking alternatives, and subsequent agreement on a 'structure' he maintains is an important part of such work."*

Teaching Mental Mathematics from Level 5 – Algebra (Page 13) gives further useful prompts to ask students to relate the observed structure of a linear sequence to the way that, and the rate at which, it grows, and from there to writing an expression for the value of the nth term.

The activity Shifting Times Tables from NRICH gives a different perspective on finding an expression for the value of the nth term of a linear sequence, by considering the link between the numbers in a sequence and the related times table:

The activity displays the terms of a linear sequence and asks the students to identify the related times table and then the shift that has been made to the times table to get the numbers in the sequence; from there an expression can be given for the value of the nth number in the shifted times table. The different levels in the activity increase in complexity: levels 1 & 2 always display consecutive terms of a linear sequence whereas levels 3 & 4 display any terms of a linear sequence.

There are bound to be different answers – not least of the "5n + 2" vs. "5n – 3" variety – and so there will be plenty of opportunity for students to present to each other and to discuss each other's suggestions. Peer challenge – and robust defence! – is a very powerful tool for developing reasoning skills, and gives many opportunities for developing the oral skills that the new Programmes of Study quite rightly highlight.

Image credit

Eyes Down

This new feature will be a regular picture-based item that you might use with your pupils, or your department, or just by yourself, to make you think about something in a different way

This picture was taken at Geneva railway station – yes, it's a holiday snap! Do you ever hear yourself saying "parallel, like train tracks"? There are lots of sets of parallel lines in this picture, but if you start to look into the distance, the train tracks, whilst remaining the same distance apart and never crossing, also start to curve as the rails go around a corner. Does "parallel" only apply to lines? Can curves be parallel? Are concentric circles parallel? If these train tracks continued dead straight for a very long distance, what would happen? Would that still count as their being parallel? And once you're thinking about spherical geometry, what, on the map, appears to be the shortest route between Heathrow Airport and Shanghai Pudong International Airport, and what, on a globe, is?

If you have a thought-inducing picture, please send a copy (ideally, about 1-2Mb), and a short (150 words maximum) commentary, to us at info@ncetm.org.uk - we look forward to hearing from you!

Image credit