MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development Materials

1 The structure of the number system

Theme overview

Guidance document | Key Stage 3

Making connections

'Teaching for mastery' describes the elements of classroom practice and school organisation that combine to give students the best chance of developing a deep, connected, embedded and sustainable understanding of mathematics.

At any one point in a student's journey through school, achieving mastery means acquiring a secure understanding of the mathematics that has been taught to enable them to move on to more advanced material.

To achieve this, students need to understand the interconnected nature of mathematics and how one idea builds on and develops from other ideas. To this end, the NCETM has identified a set of six 'mathematical themes' within Key Stage 3 mathematics that bring together a group of connected ideas or 'core concepts'.

The theme *The structure of the number system* covers the following interconnected core concepts:

1.1 Place value, estimation and rounding

1.2 Properties of number

1.3 Ordering and comparing

1.4 Simplifying and manipulating expressions, equations and formulae

Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Why is this mathematical theme important?

Students who have a deep understanding of the structure of numbers, who can partition numbers, recombine them and represent them in different ways, are more fluent and flexible when it comes to calculating (for example, when recognising that the calculation 12.5×48 can be transformed into $12.5 \times 8 \times 6$ or 100×6, a quick solution can be found). Students can use standard algorithms proficiently, because they understand why the algorithms work, but are not hindered by an exclusive or over-reliance on them when other strategies are more appropriate or efficient (e.g. $2.3 \times 7 + 3 \times 2.3$ is more easily – and elegantly – solved by noticing that 2.3 is common to both partial products and the whole expression is equal to 10×2.3 or 23).

As students learn about the mathematical structures that underpin numbers and the number system, the need to generalise these structures will arise. This` can be recorded using algebraic symbols. For instance, the $2.3 \times 7 + 3 \times 2.3$ example can be written more generally as $7a + 3a = 10a$), with the letter symbol representing a generalised number.

Key underpinning knowledge

Several important considerations are key to students gaining a secure and deep understanding of the mathematics within this theme, namely:

- that number can be represented and conceptualised in different ways:
  - 'number as object' – using materials that show number as an amount. Dienes apparatus is an example of representing number as object
  - 'number as measure' – a number line showing number as length; or the array/area model, where number as length and number as area are both present
  - 'number as symbol' – place-value counters and the Gattegno chart are operating more symbolically with numbers
- that the same number can be symbolically expressed in different ways and that different symbolisations may reveal different structures; for example, writing a number in base-ten place-value notation (e.g. 48) and writing a number as its unique prime factorisation (e.g. $2^4 \times 3$) reveal different structures of the number
- that when different ways of writing numbers are explored, it is useful to generalise using algebraic notation and, through doing this, some important conventions and aspects of algebraic notation can be learnt as an integral part of number work
- that appreciating where numbers are on a number line can support students' understanding of approximation and rounding
- that measures are an important context through which students gain familiarity with, and a deep understanding of, number, particularly decimals.

Statements of knowledge, skills and understanding

Each of the four core concepts within the theme *The structure of the number system* has been broken down further into a set of statements of knowledge, skills and understanding, as listed below.

1.1 Place value, estimation and rounding

- 1.1.1 Understand the value of digits in decimals, measure and integers
- 1.1.2 Round numbers to a required number of decimal places
- 1.1.3 Round numbers to a required number of significant figures
- 1.1.4 Estimate calculations by rounding

1.2 Properties of number

- 1.2.1 Understand multiples
- 1.2.2 Understand integer exponents and roots
- 1.2.3 Understand and use the unique prime factorisation of a number

1.3 Ordering and comparing

- 1.3.1 Work interchangeably with terminating decimals and their corresponding fractions
- 1.3.2 Compare and order positive and negative integers, decimals and fractions
- 1.3.3 Interpret and compare numbers in standard form $A \times 10^n$, $1 \leq A < 10$

1.4 Simplifying and manipulating expressions, equations and formulae

- 1.4.1 Understand and use the conventions and vocabulary of algebra including forming and interpreting algebraic expressions and equations
- 1.4.2 Simplify algebraic expressions by collecting like terms to maintain equivalence
- 1.4.3 Manipulate algebraic expressions using the distributive law to maintain equivalence
- 1.4.4 Find products of binomials
- 1.4.5 Rearrange formulae to change the subject

We have produced guidance documents that offer an overview of each core concept, as well as an overview of the content of each statement of knowledge, skills and understanding. We have also broken down each of the latter into a series of key ideas to support planning, with some of the key ideas exemplified as to what teaching for mastery may look like.

We make no suggestion that each key idea represents a lesson. Rather, the fine-grained distinctions we offer in these key ideas are intended to help you think about the learning journey irrespective of the number of lessons taught.

Not all key ideas are of equal weight and the amount of classroom time required for them to be mastered will vary, but each step is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

These materials are designed for teachers to use collaboratively when planning how they will teach for a secure and deep understanding of mathematics throughout Key Stage 3. They are underpinned by a clear set of pedagogical principles and practices.

The structure of the number system core concept guidance documents[1] can be downloaded from the NCETM website.

Links to the national curriculum

A mapping[2] of all statements of knowledge skills and understanding to the national curriculum Key Stage 3 programme of study is available on the NCETM website.

Previous learning

From Upper Key Stage 2, students will bring experience of:

- reading, writing, ordering and comparing numbers up to 10 000 000 and determining the value of each digit
- rounding any whole number to a required degree of accuracy
- using negative numbers in context
- identifying the value of each digit in numbers given to three decimal places and multiplying and dividing numbers by 10, 100 and 1 000, giving answers up to three decimal places
- using, reading, writing and converting between standard units, converting measurements of length, mass, volume and time from a smaller unit of measure to a larger unit and vice versa, using decimal notation up to three decimal places
- using symbols and letters to represent variables and unknowns in mathematical situations that they already understand, such as:
  - missing numbers, lengths, coordinates and angles
  - formulae in mathematics and science.

Future learning

In Key Stage 4, students will build on the core concepts in this mathematical theme to:

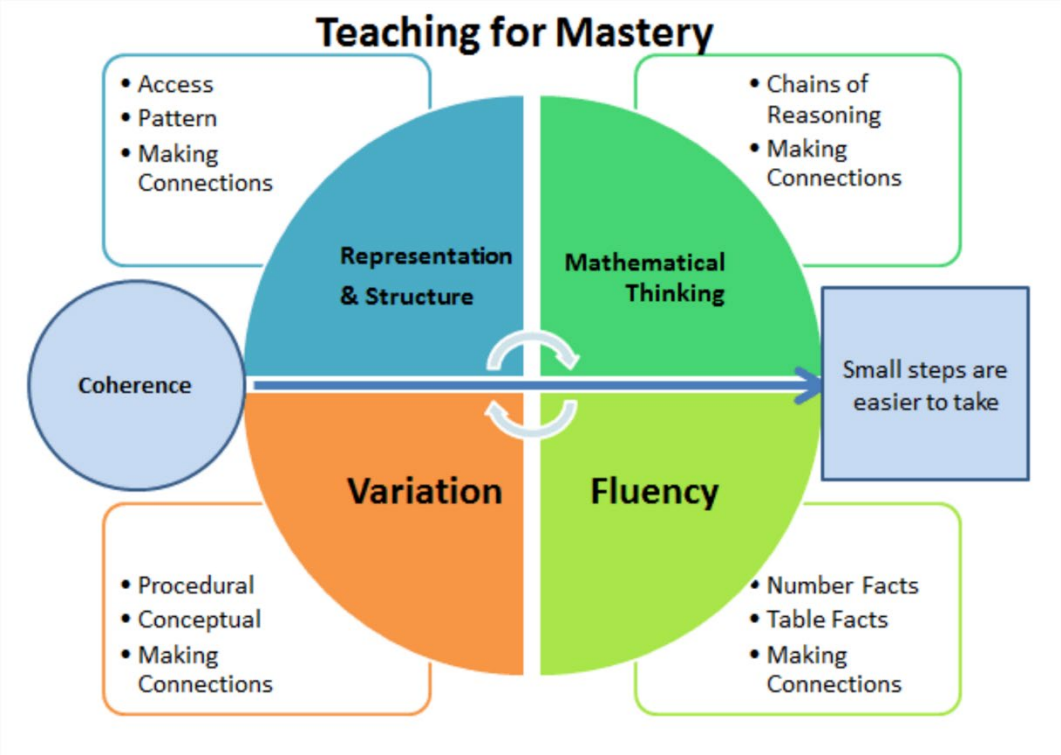
- {estimate powers and roots of any given positive number}
- calculate with roots, and with integer {and fractional} indices
- {simplify surd expressions involving squares [e.g. $\sqrt{12} = \sqrt{(4 \times 3)} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$] and rationalise denominators}
- calculate with numbers in standard form $A \times 10^n$, where $1 \leq A < 10$ and n is an integer
- {change recurring decimals into their corresponding fractions and vice versa}
- apply and interpret limits of accuracy when rounding or truncating, {including upper and lower bounds}.

Please note: Braces { } indicate additional mathematical content to be taught to more highly attaining students. Square brackets [ ] indicate content schools are not required to teach by law.

Teaching for mastery

A central component in the NCETM/Maths Hubs programmes to support the development of teaching for mastery has been discussion of Five Big Ideas[3] underpinning teaching for mastery. These are:

- Coherence
- Representation and structure
- Variation
- Fluency
- Mathematical thinking

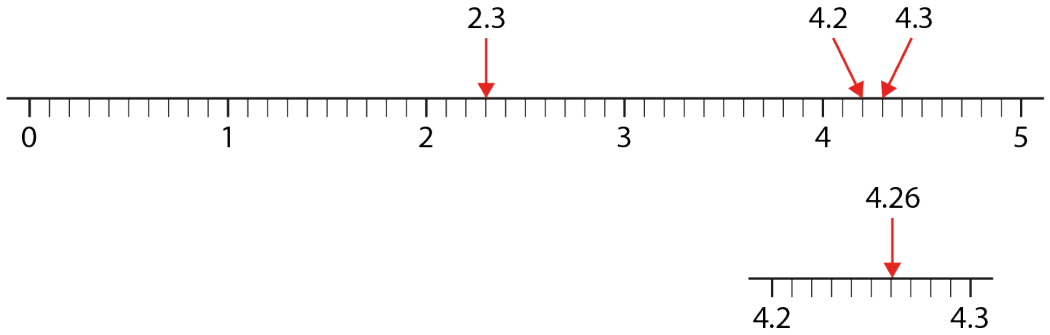
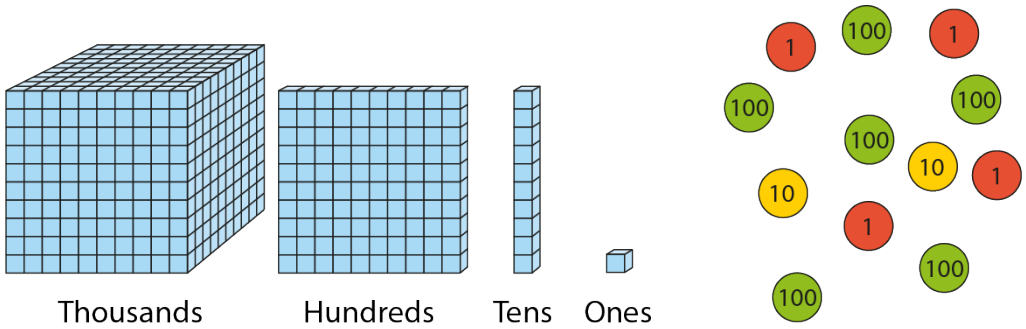


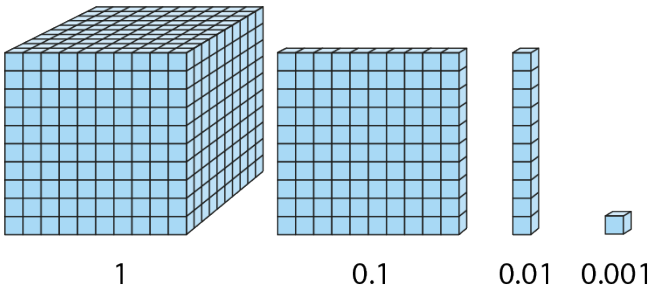
The sections below offer guidance about how these ideas relate to *The structure of the number system*.

Coherence

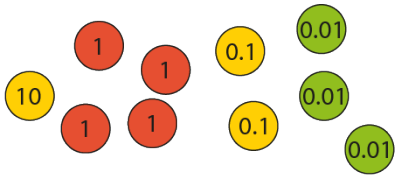
It is important to find a balance between focusing on important elements of this theme where it is useful to plan a coherent set of small steps (for example, when introducing standard form, what ideas to introduce first, what should come next, and so on) and appreciating how each idea is connected to others in the theme. For example, when thinking about numbers such as 314.2, discussing its component parts to get an understanding of the structure (for example, *'How many tens are there: 1, 31, 31.4, 31.42?'* and *'How many hundreds are there: 3, 3.1, 3.14, 3.142?'*) can lead to representing it in standard form as 3.142×10^2.

Representation and structure

Representations	Structural understanding
Number lines	Number lines provide a powerful visual image for the extension of the number system into decimals. By sub-dividing a number line into tenths, hundredths, etc., the system by which decimal numbers are named can become apparent.  Students can be encouraged to think about where numbers such as 2.3 and 4.26 might be placed in order to develop an awareness of the fractal nature of the number line.
Dienes base-ten blocks Place-value counters	 Dienes apparatus and place-value counters will probably have been an important representation used in Key Stage 2 to support students' understanding of the base-ten structure of numbers (both integers and decimals). In Key Stage 3, students are required to generalise this structure and to become as fluent with the base-ten structure of decimals as they are with integers. Re-visiting these familiar manipulatives with an emphasis on tenths, hundredths, and so on, will support this generalisation. While both Dienes and place-value counters can be used to show the place-value structure of integers and decimal numbers, Dienes can also be used to show the relative size of numbers (for example, by using the large cube to represent 1, the 'flat' to represent 0.1, the 'long' to represent 0.01 and the small cube to represent 0.001, as shown below) and to reveal the 'multiply by 10' and 'divide by 10' relationships:



When the relative size of the digits in a decimal number is understood, place-value counters can support students in manipulating and partitioning decimals in a more abstract way. For example, students could represent 14.23 as:



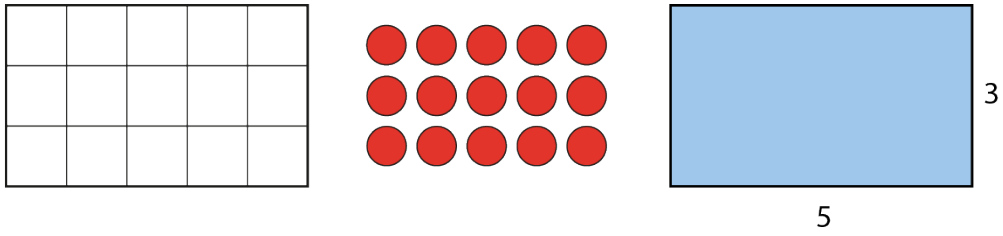
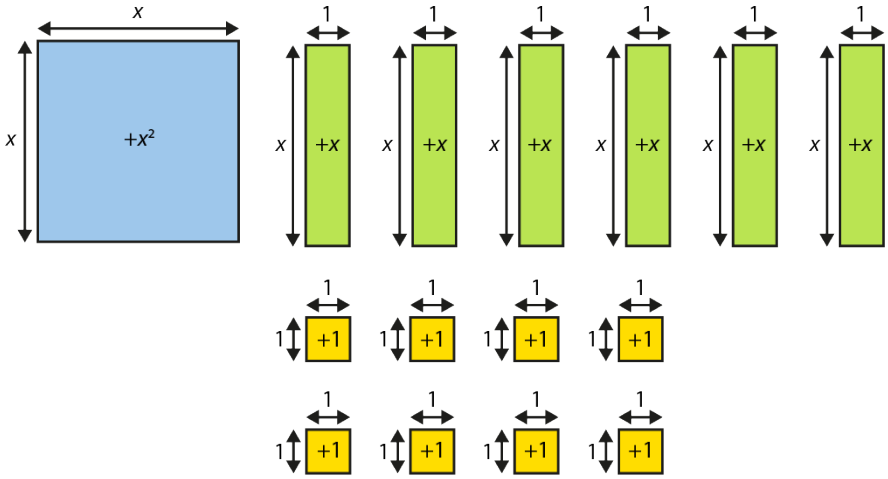
Gattegno chart

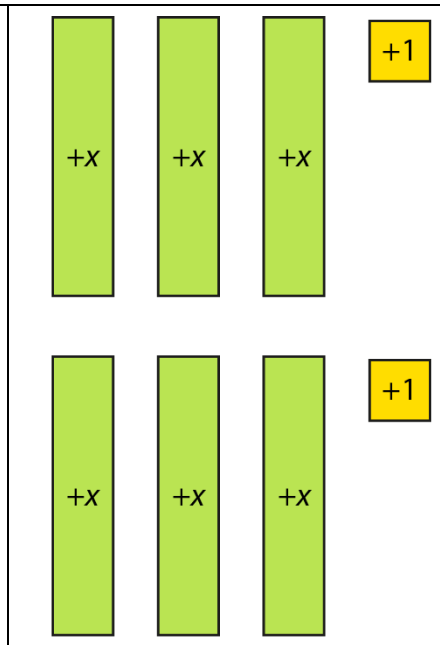
1 000	2 000	3 000	4 000	5 000	6 000	7 000	8 000	9 000
100	200	300	400	500	600	700	800	900
10	20	30	40	50	60	70	80	90
1	2	3	4	5	6	7	8	9
0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09

Rehearsing the naming of numbers (integers and decimals) by pointing to their constituent parts on the Gattegno chart, helps students to appreciate the structure of the numbers.

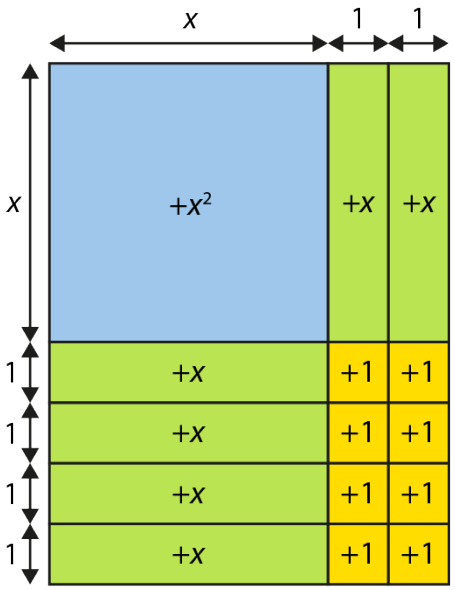
Moving (up or down) from one row to another to represent multiplying and dividing by powers of ten helps students to understand these operations and their results.

The NRICH article 'Activities on the Gattegno Chart'[4] by Alf Coles provides further details of how the Gattegno chart might be used.

Arrays and other area models	 Arrays can help students think about: • factors (for example, *'Determine the factors of 12 by finding all the rectangular arrays you can make with 12 counters.'*) • square numbers (for example, *'What numbers of counters can you arrange to make square arrays?'*) • prime numbers (for example, *'What numbers can only be made with a 1 × n (n > 1) rectangular array?'* i.e. a single line of counters).
Algebra tiles	 Although not offering a generalised image of a variable (*x* is represented by an actual length and will necessarily be seen as a particular length relative to the '1'), algebra tiles can provide a useful representation for expressions and give meaning to certain symbolic manipulations. For example:



$2(3x + 1) = 6x + 2$



$x^2 + 6x + 8 = (x + 2)(x + 4)$

Further guidance on using representations[5] in Key Stage 3 is available on the NCETM website.

Variation

Three aspects of variation that can be usefully employed:

1. Careful **choice of exercises** to 'home in' on the important concept. For example, when working on rounding numbers to a number of decimal places or significant figures, choosing examples that draw attention to the importance of 'half-way' (5, 0.5, 0.05, and so on).
2. Careful **choice of examples** to include *'what it is'* (using non-standard as well as standard examples) and *'what it is not'*. For example, when offering examples (or when inviting students to offer examples) of numbers in standard form, discussing why $657 = 0.657 \times 10^3$ or $0.00345 = 0.345 \times 10^{-2}$ are not in standard form.
3. Rather than focusing on the answer and asking only that students solve a problem, inviting students to see **in how many different ways they can solve a problem** can prompt important discussions about methods and processes, and support students' development of increasingly efficient, creative and elegant approaches. For example, when finding the prime factorisation of a number by dividing by successive prime numbers, to do this in more than one way in order to fully appreciate that such a sequence of divisions can be done in any order. This will support students' development of increasingly efficient, creative and elegant approaches.

Fluency

A key aspect of fluency is the ability to choose the quickest and most efficient strategy for a problem or calculation, which often relies on being able to represent numbers in different ways. For example, when testing to see if a number is prime, understanding why it is only necessary to try prime numbers as divisors and to only test those prime numbers up to the square root of the number.

Mathematical thinking

Throughout all the work that falls within *The structure of the number system*, the emphasis is on understanding the properties of number (decimals, fractions, powers, roots, primes, and so on). It is vital that students are prompted to reason, explain, conjecture and prove through carefully planned teacher–student and student–student discussion and not merely to listen to and follow carefully constructed teacher demonstrations and explanations. For example, when learning about factors, the following questions could be discussed and explored:

- *'Most numbers have an even number of factors. Can you explain why?'*
- *'Find some numbers which have an odd number of factors.'*
- *'Can you find a number which has exactly seven factors?'*

Further reading

NCETM secondary assessment materials[6]
Exemplar questions, tasks and activities, which may be used to support teaching and assessment. The assessment materials are mapped against the key mathematical skills and concepts within the national curriculum Key Stage 3 programme of study. Of particular relevance to *The structure of the number system* are the sections focusing on: place value, estimation and rounding (pages 5–6); ordering and comparing the value of numbers expressed in different ways (pages 7–8); properties of number (page 9); terminology and notation (pages 14–15) and simplifying and manipulating expressions, equations and formulae (pages 16–17).

Weblinks

1 Theme 1: *The structure of the number system* – core concept guidance documents
https://www.ncetm.org.uk/resources/53530

2 NCETM Key Stage 3 mastery curriculum structure, including national curriculum mapping
https://www.ncetm.org.uk/secondarymasterypd#curriculum_structure

3 Five Big Ideas in Teaching for Mastery
https://www.ncetm.org.uk/resources/50042

4 NRICH: 'Activities on the Gattegno Chart' article
https://nrich.maths.org/10741

5 Representations in Key Stage 3 – guidance documents
https://www.ncetm.org.uk/resources/53609

6 NCETM secondary assessment materials
https://www.ncetm.org.uk/resources/51246