MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

6 Geometry

6.1 Geometrical properties

Guidance document | Key Stage 3

Making connections

The NCETM has identified a set of six 'mathematical themes' within Key Stage 3 mathematics that bring together a group of 'core concepts'.

The sixth of these themes is *Geometry*, which covers the following interconnected core concepts:

6.1 **Geometrical properties**

6.2 Perimeter, area and volume

6.3 Transforming shapes

6.4 Constructions

This guidance document breaks down core concept *6.1 Geometrical properties* into three statements of knowledge, skills and understanding:

6.1.1 Understand and use angle properties

6.1.2 Understand and use similarity and congruence

6.1.3 Understand and use Pythagoras' theorem

Then, for each of these statements of knowledge, skills and understanding we offer a set of key ideas to help guide teacher planning.

Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Overview

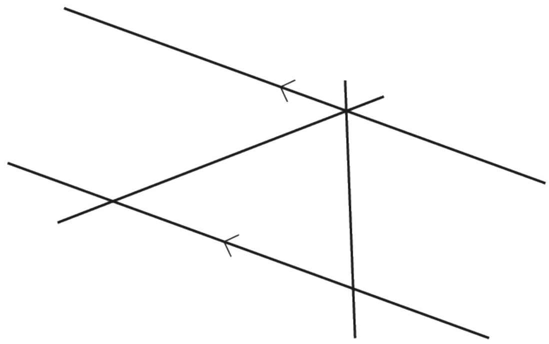
A 2001 report about the teaching and learning of geometry from 11–19 (The Royal Society / Joint Mathematical Council working group, 2001: xii, 7)† set out a number of objectives for a well-taught geometry curriculum, notably:

- to develop spatial awareness, geometrical intuition and the ability to visualise
- to encourage the development and use of conjecture, deductive reasoning and proof.

Students will have had opportunities to develop their spatial awareness and geometrical intuition in Key Stage 2 through situations involving angles (angles meeting at a point, angles on a straight line, vertically opposite angles and angles in regular polygons) and similar shapes. They will be aware of the geometrical facts and properties inherent in these situations, and an important development throughout Key Stage 3 is to be able to reason and construct proofs for why such facts and properties hold.

Throughout this core concept, the emphasis is on understanding *why* certain facts are true and *why* certain properties hold.

In the context of angles, the geometry of intersecting lines and the connections and deductions that can be made from diagrams, such as this:



provide rich opportunities in which students can be encouraged to build logical, deductive arguments. Students develop a narrative, connecting and combining known facts in order to generate further mathematical truths.

In studying similarity and congruence, students are required to go beyond intuitively recognising when shapes are similar or congruent, and to think about what can change and what has to stay the same for these properties to hold.

While learning about an important theorem in mathematics, such as Pythagoras' theorem, there is an opportunity to go beyond *knowing that* it is true to *knowing why* and appreciating the beauty of discovery.

Teaching and learning associated with this core concept offers an opportunity for students to think about relationships and structures, to reason with them and to prove results. As such, it helps them form

† The Royal Society / Joint Mathematical Council working group, 2001, *Teaching and learning geometry 11–19*, London: The Royal Society

a view of mathematics (a view central to true mastery of the subject) that is so much more than memorising facts and practising procedures.

Geometrical properties, possibly above all other areas of mathematics, offers students a set of contexts with which to build their understanding of key mathematical concepts and the nature of mathematics itself.

Prior learning

Before beginning to teach *Geometrical properties* at Key Stage 3, students should already have a secure understanding of the following from previous study:

Key stage	Learning outcome
Upper Key Stage 2	• Recognise angles where they meet at a point, are on a straight line, or are vertically opposite, and find missing angles • Compare and classify geometric shapes based on their properties and sizes and find unknown angles in any triangles, quadrilaterals, and regular polygons • Solve problems involving similar shapes where the scale factor is known or can be found
Key Stage 3	• 1.4.1 Understand and use the conventions and vocabulary of algebra including forming and interpreting algebraic expressions and equations • 1.4.2 Simplify algebraic expressions by collecting like terms to maintain equivalence • 1.4.3 Manipulate algebraic expressions using the distributive law to maintain equivalence • 1.4.4 Find products of binomials • 1.4.5 Rearrange formulae to change the subject **Please note:** Numerical codes refer to statements of knowledge, skills and understanding in the NCETM breakdown of Key Stage 3 mathematics.

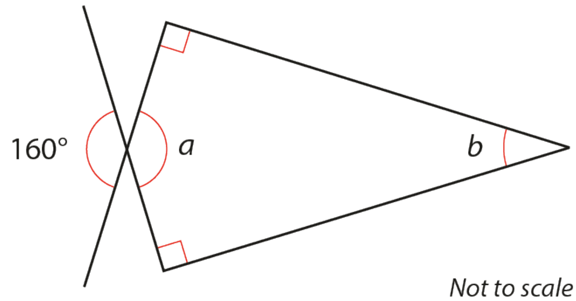
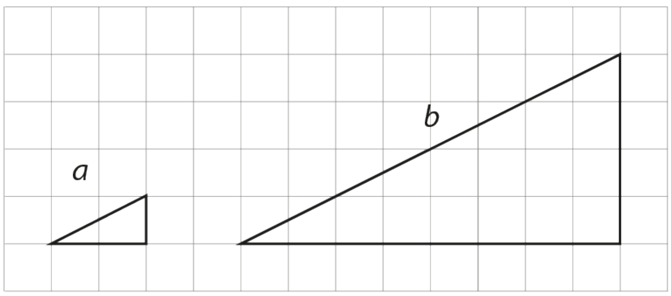
You may find it useful to speak to your partner schools to see how the above has been covered and the language used.

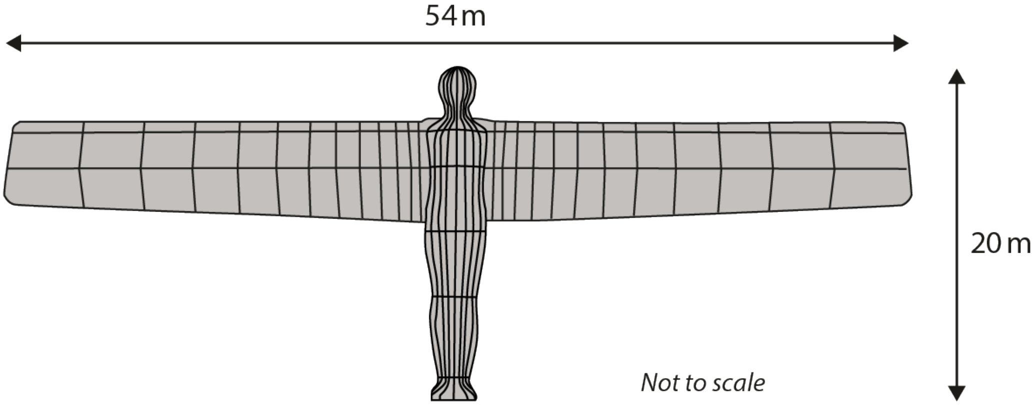
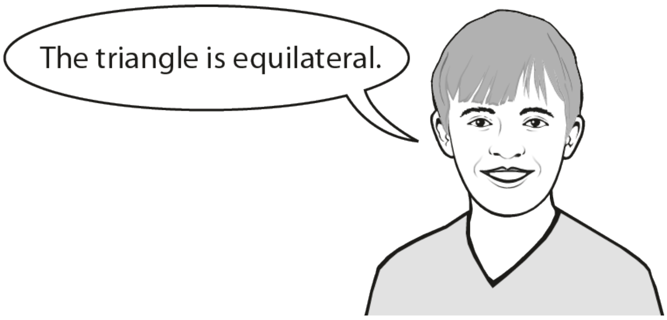
You can find further details regarding prior learning in the following segment of the NCETM primary mastery professional development materials[1]:

- Year 6: 2.27 Scale factors, ratio and proportional reasoning

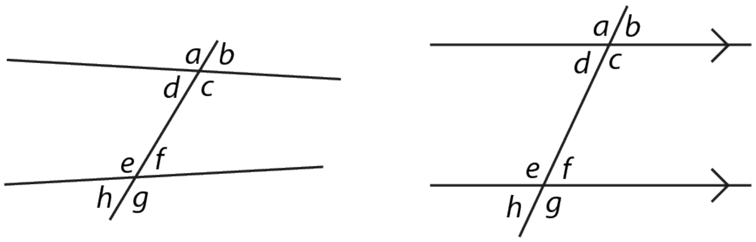
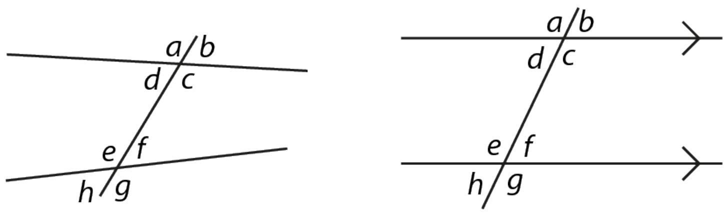
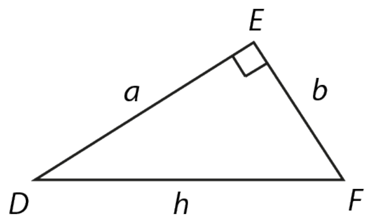
Checking prior learning

The following activities from the Standards & Testing Agency's past mathematics papers[2] offer useful ideas for assessment, which you can use in your classes to check whether prior learning is secure:

Reference	**Activity**
2016 Key Stage 2 Mathematics Paper 2: reasoning Question 17	*Calculate the size of angles a and b in this diagram.*  *Not to scale* Source: *Standards & Testing Agency* Public sector information licensed under the Open Government Licence v3.0
2017 Key Stage 2 Mathematics Paper 2: reasoning Question 22	*Here are two similar right-angled triangles.*  *Write the ratio of side a to side b.* Source: *Standards & Testing Agency* Public sector information licensed under the Open Government Licence v3.0

2018 Key Stage 2 Mathematics Paper 3: reasoning Question 9	*The Angel of the North is a large statue in England.* *It is 20 metres tall and 54 metres wide.*  *Ally makes a scale model of the Angel of the North.* *Her model is 40 centimetres tall.* *How **wide** is her model?* Source: *Standards & Testing Agency* Public sector information licensed under the Open Government Licence v3.0
2018 Key Stage 2 Mathematics Paper 3: reasoning Question 14	*Two of the angles in a triangle are 70° and 40°* *Jack says,*  *Explain why Jack is **not** correct.* Source: *Standards & Testing Agency* Public sector information licensed under the Open Government Licence v3.0

Key vocabulary

Term	Definition
alternate angles	Where two straight lines are cut by a third, as in the diagrams, the angles *d* and *f* (also *c* and *e*) are alternate. Where the two straight lines are parallel, alternate angles are equal. 
congruent (figures)	Two or more geometric figures are said to be congruent when they are the same in every way except their position in space. Example: Two figures, where one is a reflection of the other, are congruent since one can be transposed onto the other without changing any angle or edge length.
corresponding angles	Where two straight-line segments are intersected by a third, as in the diagrams, the angles *a* and *e* are corresponding. Similarly, *b* and *f*, *c* and *g*, and *d* and *h* are corresponding. Where parallel lines are cut by a straight line, corresponding angles are equal. 
hypotenuse	In trigonometry, the longest side of a right-angled triangle. The side opposite the right-angle.
interior angle	At a vertex of a polygon, the angle that lies within the polygon.
Pythagoras' theorem	In a right-angled triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other sides, i.e. the sides that bound the right angle. Example:  When $\angle DEF$ is a right angle, $a^2+b^2=h^2$.

similar	Two geometrical objects are said to be similar if they both have the same shape. More precisely, one can be obtained from the other by uniformly scaling (enlarging or reducing), possibly with additional translation, rotation and reflection. Similar shapes have corresponding sides proportional and corresponding angles equal.
supplementary angle	Two neighbouring angles whose sum is 180°. When two lines intersect each other, the resulting adjacent angles are supplementary.
transversal	A line that cuts across two or more (usually parallel) lines.

Collaborative planning

Below we break down each of the three statements within *Geometrical properties* into a set of key ideas to support more detailed discussion and planning within your department. You may choose to break them down differently depending on the needs of your students and timetabling; however, we hope that our suggestions help you and your colleagues to focus your teaching on the key points and avoid conflating too many ideas.

Please note: We make no suggestion that each key idea represents a lesson. Rather, the 'fine-grained' distinctions we offer are intended to help you think about the learning journey irrespective of the number of lessons taught. Not all key ideas are equal in length and the amount of classroom time required for them to be mastered will vary, but each is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

The following letters draw attention to particular features:

D Suggested opportunities for **deepening** students' understanding through encouraging mathematical thinking.

L Examples of shared use of **language** that can help students to understand the structure of the mathematics. For example, sentences that all students might say together and be encouraged to use individually in their talk and their thinking to support their understanding (for example, *'The smaller the denominator, the bigger the fraction.'*).

R Suggestions for use of **representations** that support students in developing conceptual understanding as well as procedural fluency.

V Examples of the use of **variation** to draw students' attention to the important points and help them to see the mathematical structures and relationships.

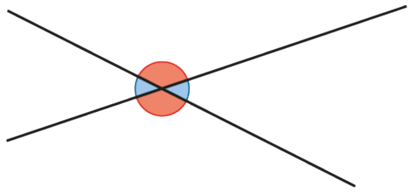
PD Suggestions of questions and prompts that you can use to support a **professional development** session.

For selected key ideas, marked with an asterisk (*), we exemplify the common difficulties and misconceptions that students might have and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches, together with suggestions and prompts to support professional development and collaborative planning. You can find these at the end of the set of key ideas.

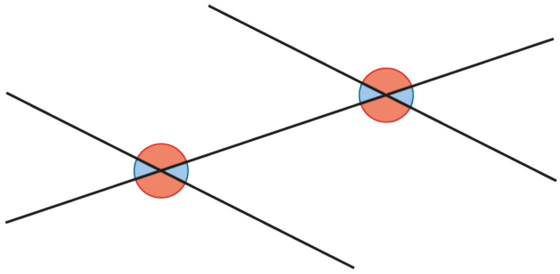
Key ideas

6.1.1 Understand and use angle properties

In Key Stage 2, students should have studied that vertically opposite angles are equal:

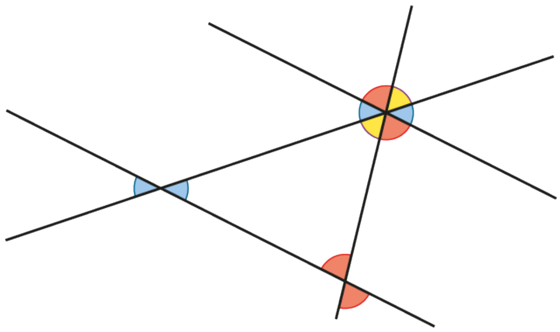


In Key Stage 3, students will use this fact to deduce that a translation of one of the lines will create a second, equivalent pair of vertically opposite angles:



They will then be able to identify pairs of equal angles (some of which are named alternate or corresponding) and other relationships, such as pairs of supplementary angles (i.e. with a sum of 180°).

Throughout Key Stage 3, students are encouraged to use what they know to construct a logical argument and deduce other facts. By offering carefully selected contexts that encourage reasoning, students can construct and understand proofs, such as the angles in a triangle always sum to 180° (see exemplified key idea 6.1.1.2):



6.1.1.1* Understand that a pair of parallel lines traversed by a straight line produces sets of equal and supplementary angles

6.1.1.2* Know and understand proofs that in a triangle, the sum of interior angles is 180 degrees

6.1.1.3 Know and understand proofs for finding the interior and exterior angle of any regular polygon

6.1.1.4 Solve problems that require use of a combination of angle facts to identify values of missing angles, providing explanations of reasoning and logic used

6.1.2 Understand and use similarity and congruence

Students will already be familiar with similarity through their work on proportional reasoning. In this set of key ideas, the focus shifts to properties that may not have been explicitly addressed before, particularly the preservation of angle size when shapes are enlarged.

When exploring congruence, students should be aware of what is changing but also what is staying the same, and investigate what changes are possible which maintain congruence. Exploring similarity and congruence with a range of polygons and triangles should help students refine their understanding of these concepts and avoid confusion between them.

In addition, exploring rotational symmetry offers students a further set of geometrical properties with which to describe and classify shapes.

6.1.2.1* Recognise that similar shapes have sides in proportion to each other but angle sizes are preserved

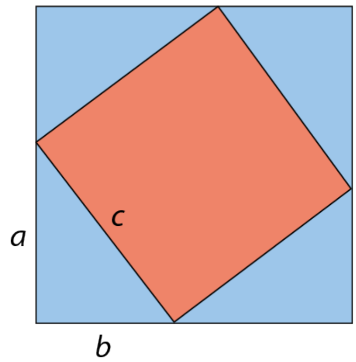
6.1.2.2 Recognise that for congruent shapes both side lengths and angle sizes are preserved

6.1.2.3 Understand and use the criteria by which triangles are congruent

6.1.2.4 Recognise rotational symmetry in shapes

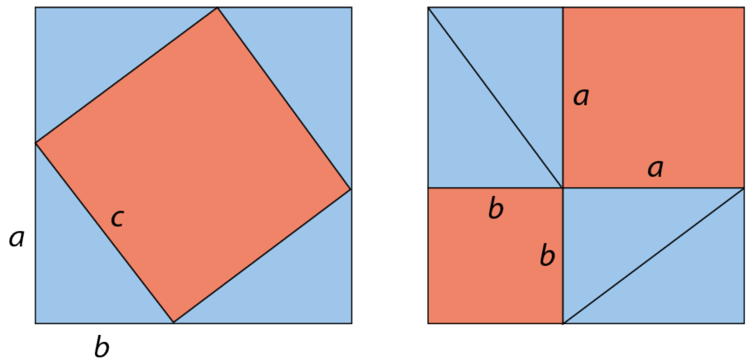
6.1.3 Understand and use Pythagoras' theorem

The relationship described by Pythagoras' theorem offers another context for students to reason deductively and use known facts to generate other mathematical truths. Offering students a diagram, such as this:



and asking them to identify known lengths and areas, can develop students' awareness of the relationship.

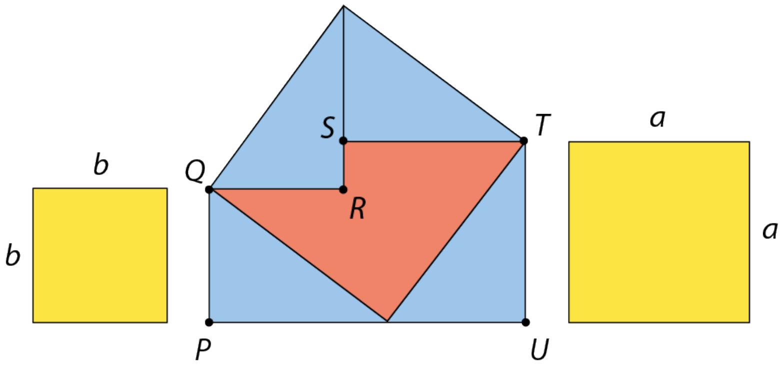
Method 1

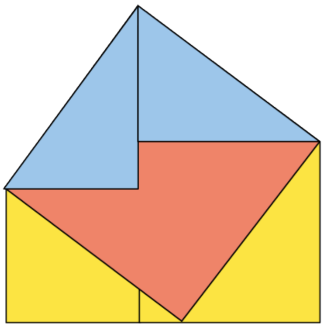


The left-hand diagram is made up of four congruent right-angled triangles arranged in a particular way inside a square. The area formed by the space enclosed by these triangles looks like a square and, by examining what is already known about the angles and sides of the triangles, it can be shown that it is a square. By rearranging the four triangles as shown in the right-hand diagram, it is possible to deduce that the area of the two smaller (red) squares must be the same as the area of the larger (red) square, resulting in the following statement about any right-angled triangle:

$$a^2 + b^2 = c^2$$

Method 2

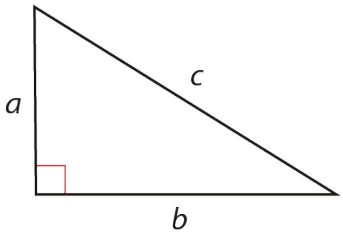


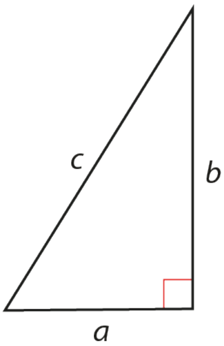


Method 2 may prompt an alternative argument leading to the same result. The left-hand diagram shows two of the triangles being subsumed into the 'tilted' square; from this, it can be deduced that the area of the tilted square (c^2) equals the area of the composite shape *PQRSTU*, and this is equivalent to the area of the two yellow squares (a^2 and b^2).

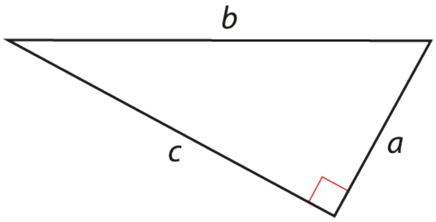
A key awareness for students is the difference between a proof and a demonstration. Students may be convinced of Pythagoras' theorem by measuring and recording their results in a table, observing the relationship, or by other demonstrations. However, their attention should be drawn to the difference between such demonstrations and a proof that depends on the geometrical structure involved.

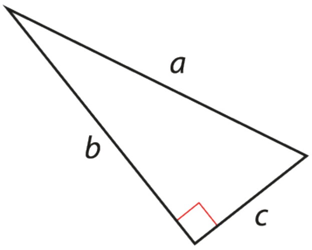
When using Pythagoras' theorem to solve problems, it will be important to include a wide range of problems (both from real-life contexts and in more abstract geometrical diagrams) so that students do not lapse into a mechanical application of $a^2 + b^2 = c^2$ without thinking about the problem. It is helpful to avoid over-use of standard examples where the right-angled triangles are always in a similar orientation, like this:



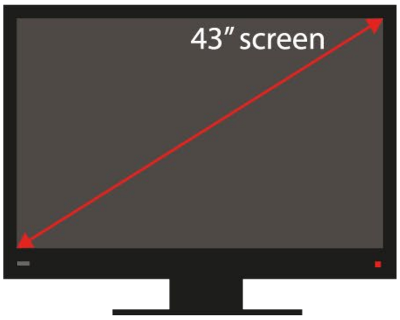


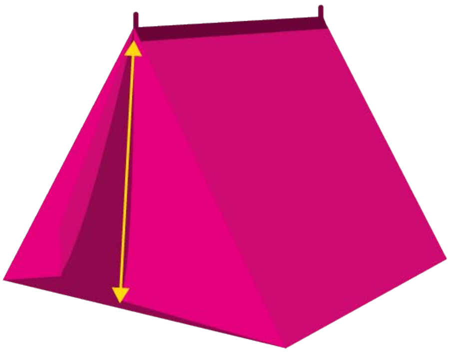
Instead, include some non-standard orientations, like this:





It is also useful to give students opportunities to look at more complex arrangements (including real-life situations) where the right-angled triangle needs to be found and isolated from the rest of the information before Pythagoras' theorem can be used. For example:



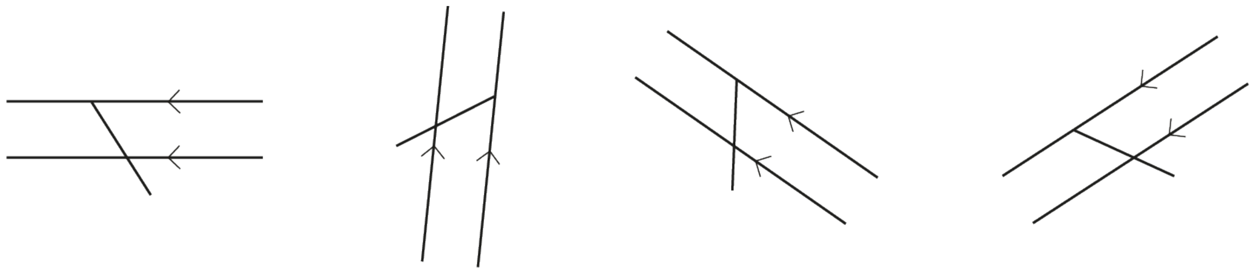
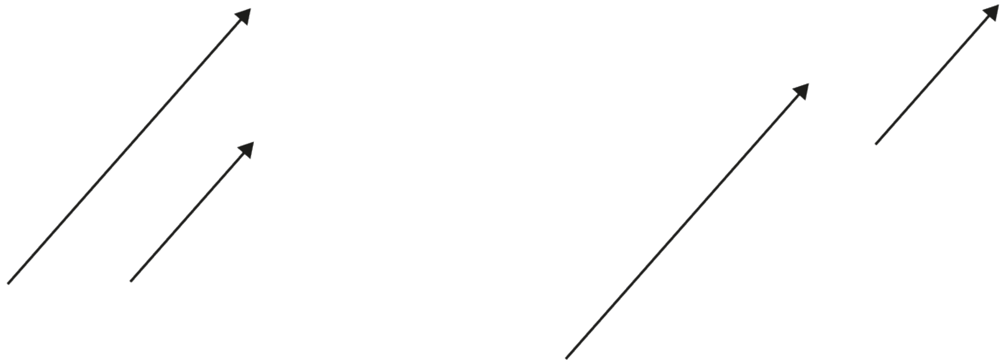


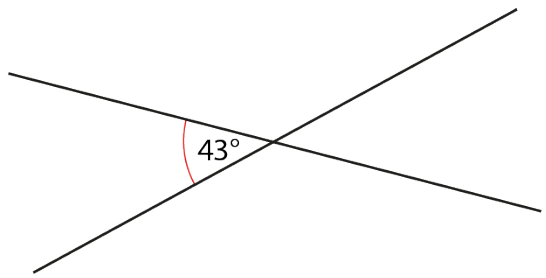
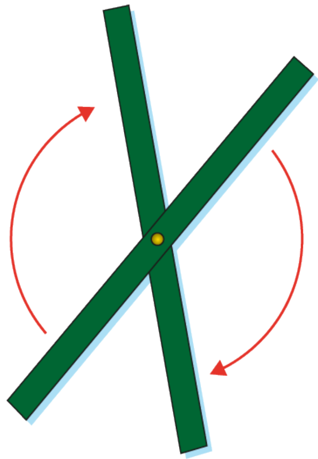
6.1.3.1 Be aware that there is a relationship between the lengths of the sides of a right-angled triangle

6.1.3.2* Use and apply Pythagoras' theorem to solve problems in a range of contexts

Exemplified key ideas

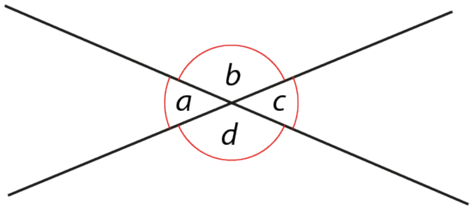
6.1.1.1 Understand that a pair of parallel lines traversed by a straight line produces sets of equal and supplementary angles

Common difficulties and misconceptions
Students often confuse alternate and corresponding angles and, consequently, may not be able to pick out examples of these in any but the simplest of diagrams and shapes. Students should be given plenty of opportunities to notice such angles in non-standard diagrams, such as these:  They should be encouraged to use reasoning based on the inherent geometrical structure rather than memorising standard diagrams and using phrases such as 'F' and 'Z' angles, which do not support reasoning and mathematical thinking. Another common difficulty is that students recognise and accept lines as being parallel only if they are of a similar length and position; off-set lines are often not perceived as being parallel, particularly when there is little overlap, as below:  **L** Encourage the use of precise language when describing angle properties. For example, use 'alternate angle' rather than 'Z angle' and state the full angle property when reasoning, e.g. *'$x = 45°$ because alternate angles are equal.'*

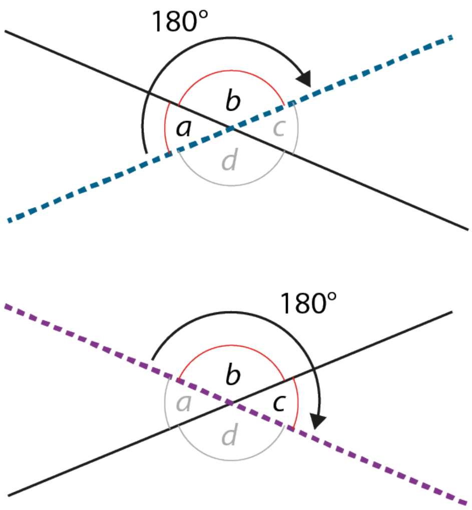
What students need to understand	Guidance, discussion points and prompts
Recognise that two intersecting lines create two pairs of equal and opposite angles. *Example 1:* *Given this one angle, can you find any other angles in the diagram?*  *Explain your reasoning.*	**R** *Example 1* provides a key representation for students to reason with to help them recognise which angles are equal, which are supplementary, and why. Modelling such a structure using two sticks or thin strips of card hinged at the point of intersection, can help students to see how, as one angle increases so does the other and that they must be equal:  **PD** How might dynamic geometry software be used to demonstrate these relationships? When might students benefit from having a physical model of vertically opposite angles and where does use of technology fit in the learning sequence?

Example 2:

Write down as many equations (or statements) as you can to show the relationships between the angles in this diagram.



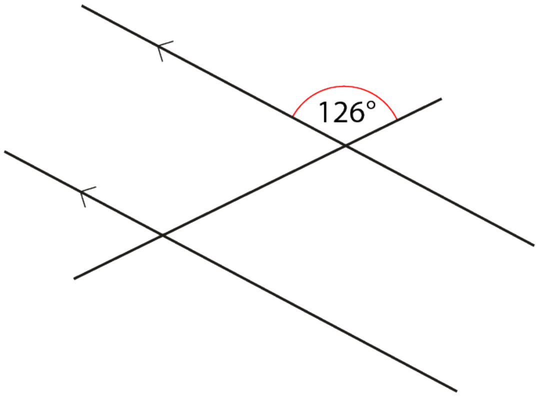
Draw attention to both pairs of equal angles and highlight both straight lines and the associated supplementary angles at each stage.



Recognise that adding another line parallel to the original creates a further two pairs of equal and opposite angles that are identical to the original four.

Example 3:

Given this one angle, can you find any other angles in the diagram?



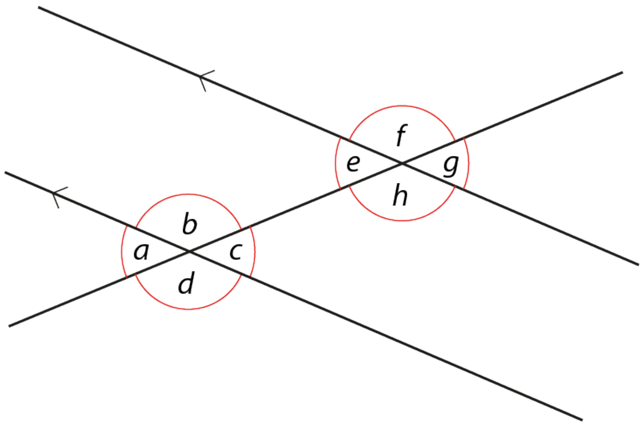
Explain your reasoning.

In *Example 3*, adding a parallel line helps students to see that the angles made are the same as the original four.

This diagram also provides an opportunity for students to notice that the lines are parallel if, and only if, the angles are equal.

Example 4:

Write as many equations (or statements) as you can to show the relationships between the angles in this diagram.

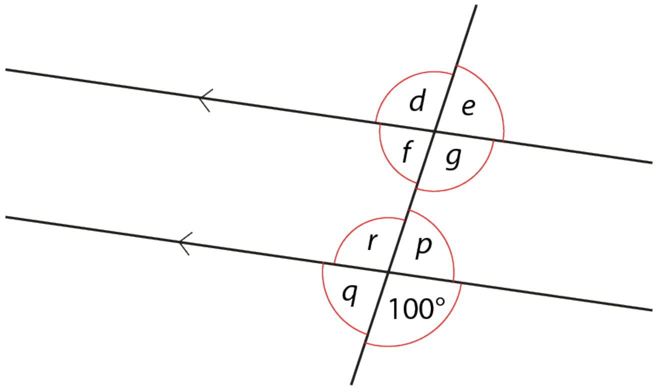


Explain your reasoning.

With *Example 4*, you could use tracing paper to show how the cluster of four angles *a*, *b*, *c* and *d* can be translated along the transversal to fit exactly on top of the *e*, *f*, *g*, *h* cluster, in order to support students' understanding.

Example 5:

David thinks angle e = 100° because corresponding angles are equal. Do you agree? Explain your answer.



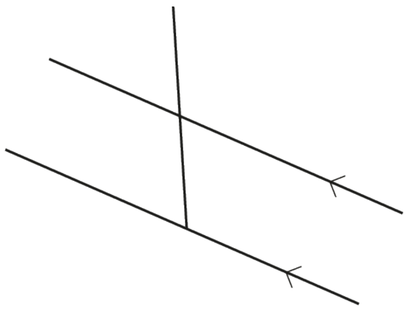
V *Example 5* gives students an opportunity to explore misconceptions. In particular, it draws attention to the common confusion between alternate and corresponding angles.

R In any diagram such as this, giving one angle and asking *'What else do you know?'* helps to draw attention to the structure of the diagram, rather than students trying to figure out how they would find a certain angle.

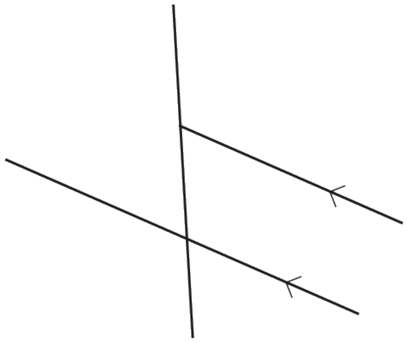
Example 6:

If the acute angle in each of these diagrams is 50°, what are the other angles?

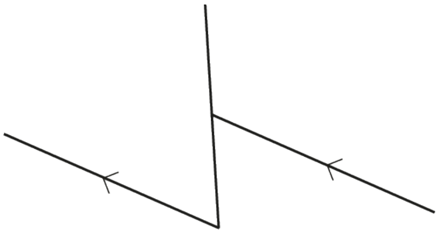
a)



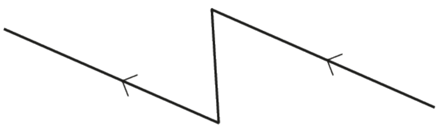
b)



c)



d)



V By systematically making small changes to a diagram, students will be able to see examples of equal and supplementary pairs of angles within the overall structure of two incomplete sets.

Attention can also be drawn to the fact that the reflex angles in parts c) and d) are made up of a combination of one of the acute and two obtuse angles.

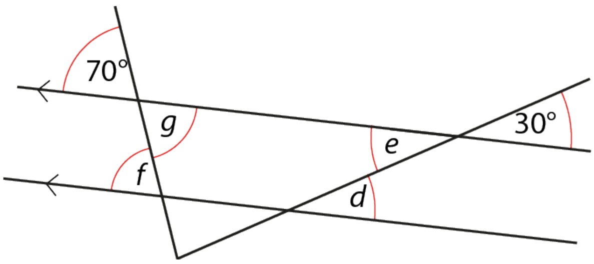
D Students could be challenged to draw a similar diagram where the reflex angle is made up of one obtuse and two acute angles.

Identify angles in more complex diagrams.

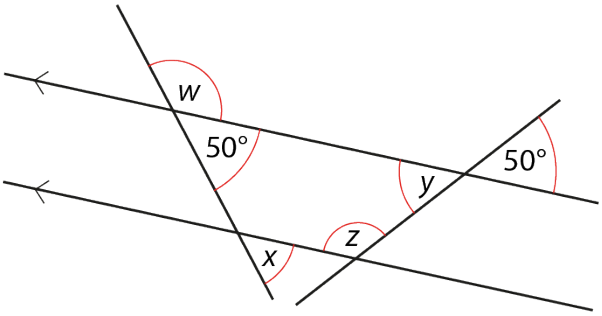
Example 7:

Find the value of the labelled angles.

a)



b)

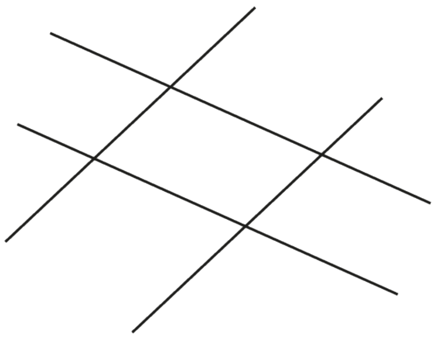


Example 7 has been designed so students use and apply their knowledge and skills of identifying corresponding, alternate, vertically opposite and supplementary angles, rather than using other properties such as the sum of interior angles in a triangle being 180°.

D Whole-class discussion based around students' responses to this example may be deepened by considering other properties. For example, in part a), attention could be drawn to the acute angle that is neither 30° nor 70° and students could discuss how this might be found. Similarly, in part b), attention could be drawn to the isosceles trapezium and deductions made about other angles, using notions of symmetry and equal length.

Example 8:

If all the acute angles in this diagram are equal, what shape is made?

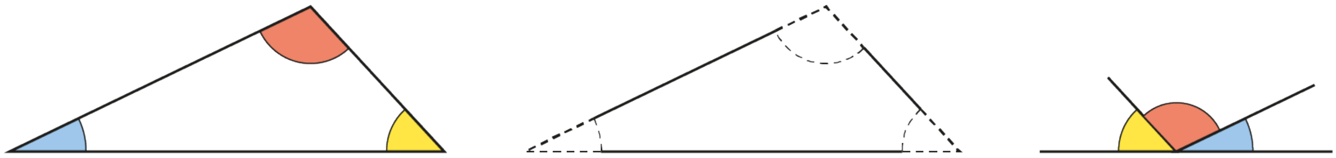


D *Example 8* offers an opportunity for students to deduce that opposite lines in this diagram have to be parallel (because all of the acute angles are the same). Therefore, this example has the potential to produce some useful discussion about the properties of a parallelogram.

6.1.1.2 Know and understand proofs that in a triangle, the sum of interior angles is 180 degrees

Common difficulties and misconceptions

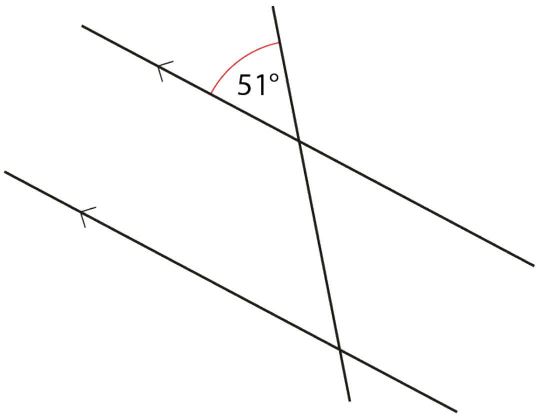
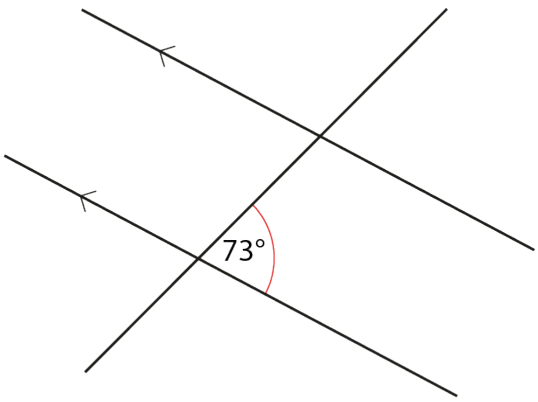
Students often confuse a demonstration for a proof. In Key Stage 2, students learnt that the sum of the interior angles of a triangle is 180° and will have used this fact to calculate missing angles. They may have seen a demonstration of this fact involving cutting out a paper triangle, tearing off the three corners and showing that the angles can be placed together on a straight line:

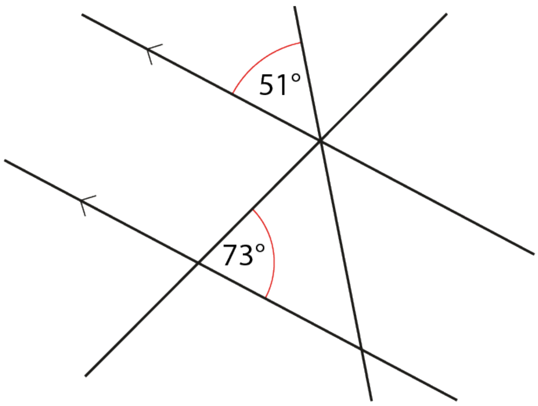
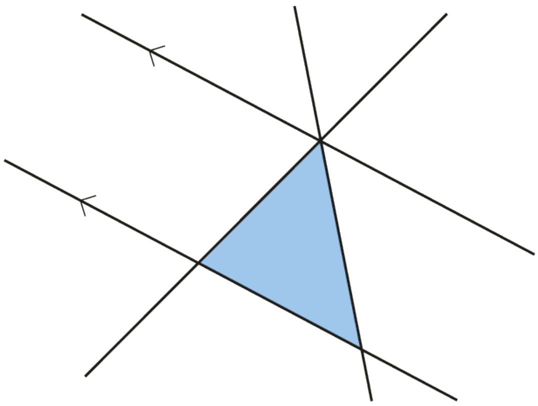


It is important that students appreciate that, while this indicates that such a relationship might be true (and indeed is a very useful way of appealing to students' intuition), it is a demonstration and not a proof.

In Key Stage 3, students will develop their understanding of what is meant by mathematical proof. This is likely to include understanding proof as a form of convincing argument based on logical deduction and an expression of generalisation, as opposed to checking against a few specific cases. Students are also developing an understanding about the conventions of communicating proof, including the use of language such as 'if … then', 'therefore' and 'because', and correct and unambiguous use of mathematical symbolism.

What students need to understand	Guidance, discussion points and prompts
Appreciate the difference between a demonstration and a proof. *Example 1:* *Emma and Samira each show that the angles in a triangle add up to 180°.* *Emma constructs a triangle using a pair of compasses and a ruler, measures each of the interior angles and adds them up. They have a sum of 180°. She repeats this for two different triangles and finds the same result.* *Samira cuts out a paper triangle, tears off all three corners and places them along the edge of a ruler to show that they fit together and lie on a straight line.* *Give reasons why Emma and Samira have not produced a convincing argument.*	**R** Consider asking students to repeat Emma and Samira's approaches. Through doing the activity themselves, students will come to appreciate the role of measurement and estimation in both methods. **D** Questions which might promote deeper thinking from students include: • *'How do we know that the sum of the angles is not 179° or 181°?'* • *'Do you think it is a coincidence that the angle sum of a triangle is exactly half a turn? Why do you think that might be?'*

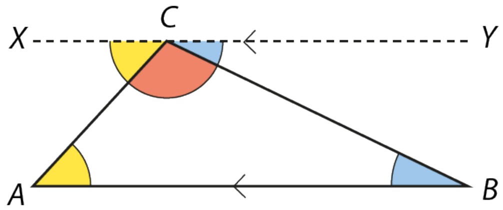
Understand that the angle sum of a triangle result is built from existing knowledge of angle facts. *Example 2:* *Look at this diagram:*  *Fill in all the missing angles.* *Give a clear explanation for each answer.*	**R** The diagram (similar to those introduced earlier in 6.1.1.1) of a pair of parallel lines with a transversal is a very powerful image. It contains within it all the structures and relationships that are needed to deduce the angle sum of a triangle. **PD** Most proofs for the angle sum of a triangle are based upon properties of angles produced from a pair of parallel lines and a transversal, yet 'angle sum of a triangle' often precedes 'alternate and corresponding angles' work within a taught curriculum. What are the fundamental geometric proofs for Key Stage 3 (and GCSE) and in what order should they be approached?
Example 3: *Look at this diagram:*  *Fill in all the missing angles.* *Give a clear explanation for each answer.*	It is important for students to be given time to discuss all the relationships they see in this diagram at each of the stages outlined in the *Examples 2*, *3* and *4*, so that they thoroughly internalise these structures.

Example 4: *This diagram is a combination of the diagrams from Examples 2 and 3:*  *a) Use your previous answers and fill in as many missing angles as you can.* *b) What do you notice about the three angles in the shaded triangle?*  *c) Where else can you see these three angles next to each other?*	By combining the two diagrams in *Examples 2* and *3*, a triangle is created. This build-up of facts, deduced from already known and understood relationships, allows students to arrive at a convincing argument about the sum of the angles in a triangle for themselves. Part c) draws attention to the relevance of 180° as half a full turn or the sum of the angles on a straight line.

Appreciate different proofs of the same result.

Example 5:

Rearrange the statements to prove that the sum of the interior angles of triangle ABC is 180°.



Statement 1

$\angle ACX = \angle CAB$ *because alternate angles are equal*

Statement 2

$\therefore \angle CAB + \angle ACB + \angle CBA = 180°$

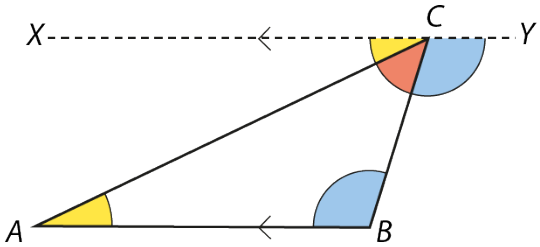
Statement 3

$\angle BCY = \angle CBA$ *because alternate angles are equal*

Statement 4

$\angle ACX + \angle ACB + \angle BCY = 180°$ *because angles on a straight line at a point add up to 180°*

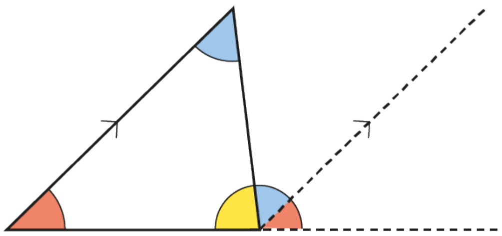
R A dynamic image of this diagram, where points can be moved, would support students in understanding the generalisability of this fact (i.e. that it is true for any triangle, not just an acute-angled one).



L Such examples, where students are asked to rearrange the statements of a proof, will support their use of correct terminology and symbolism. This provides a useful scaffolding device to prepare students for constructing their own reasoned arguments and proofs (both verbally and in written form) in the future.

Example 6:

a) *Look at this diagram and think what it might be showing.*



b) *Share your ideas with your partner and agree what the diagram is telling you.*

c) *Work with your partner to construct a clear written argument which explains (using the diagram) why the angle sum of a triangle is 180°.*

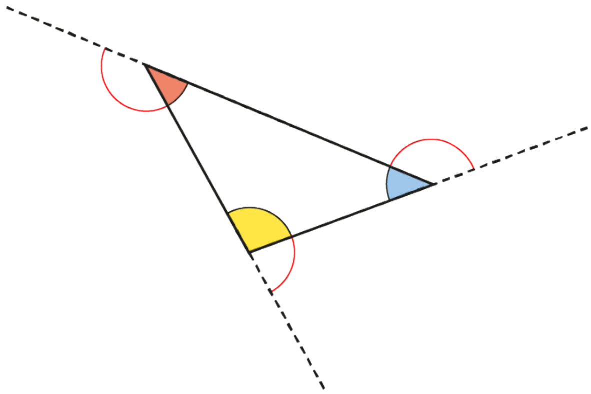
R Consider having a large version of the diagram from *Example 6* displayed so that everyone can see it clearly.

Give students time to look at the diagram and, on their own, make sense of it before sharing their ideas with their partner.

L It will be useful to share as a whole class (using a visualiser or asking students to show their work on the board) how students have expressed themselves. It will be important to share examples of correct use of mathematical language (for example, appropriate use of 'alternate' and 'corresponding' angles) as well as clear and efficient use of mathematical symbolism.

When students can clearly articulate what they want to say using precise mathematical language and symbolism, they have a tool for thinking clearly and logically.

PD What other opportunities might you offer students to construct clear written and spoken arguments and proofs?

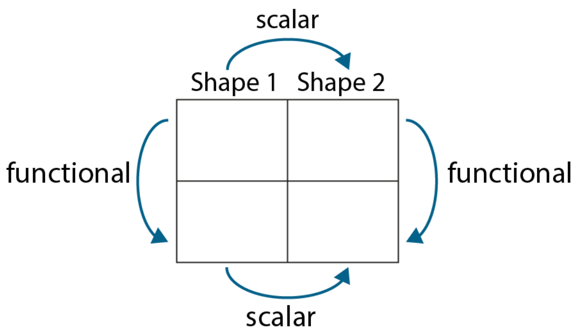
Example 7: *Given that the exterior angles of a polygon add up to 360°, prove that the interior angles of this triangle add up to 180°.* 	**R** It may be helpful to begin *Example 7* by drawing a triangle on the floor of the classroom (maybe on a large piece of sugar paper). Get students to walk around the perimeter, taking note of the direction they are facing when they start, the turns they make at each vertex, and the direction they are facing when they return to their starting point. Also, by using a version of this diagram on a computer or interactive whiteboard, you can show how it can be scaled down proportionately to show how the triangle reduces to a point and the external angles clearly show a full turn.

6.1.2.1 Recognise that similar shapes have sides in proportion to each other but angle sizes are preserved

Common difficulties and misconceptions

Students might intuitively recognise similar and congruent shapes without fully appreciating what can change and what must stay the same for these properties to hold. It is important for students to go beyond their intuition and think deeply about the multiplicative relationships connecting the sides of similar shapes. Some students may have difficulties recognising the multiplicative structure and finding the multipliers, so resort to additive methods (see *Examples 3* and *4*). Some students may also not appreciate angle preservation in similar shapes (see *Examples 6* and *7*).

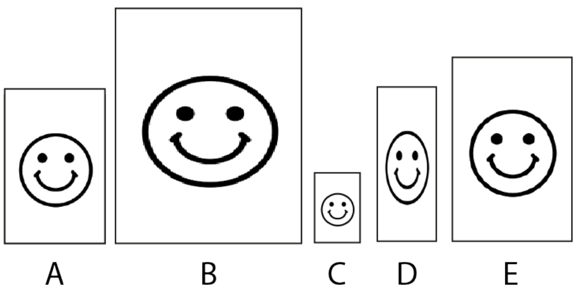
R A ratio table is a powerful representation to explore and find the scalar and functional multipliers in similar shapes.



This is explored further in *Examples 4* and *5* (ratio tables are also discussed in greater depth in key idea *3.1.2.2* in *3.1 Understanding multiplicative relationships*).

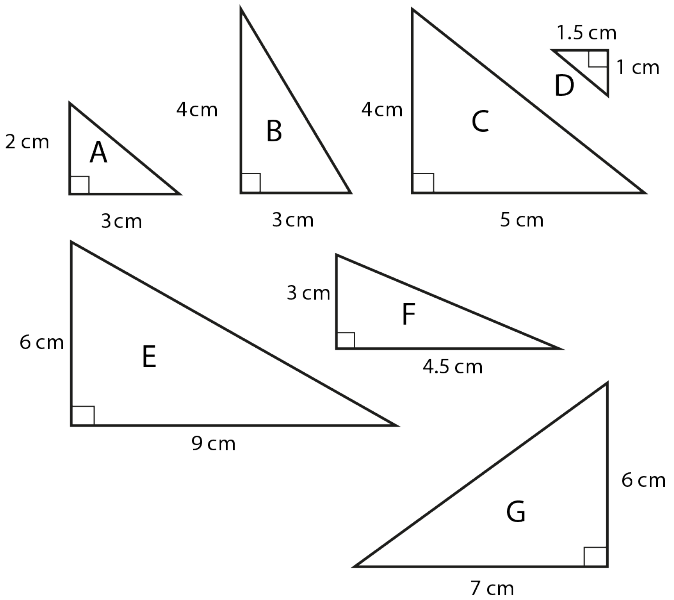
Some students have difficulties with the concept of similarity as they use the dictionary definition of the word 'similar' – shapes with a resemblance in appearance without being identical – rather than the mathematical definition.

L Similar shapes have corresponding sides proportional and corresponding angles equal.

What students need to understand	Guidance, discussion points and prompts
Appreciate the relationship between corresponding sides in similar shapes is multiplicative *Example 1:* *Which photos are similar?* 	**V** The images in *Example 1* are designed to prompt discussions about what can change and what must stay the same for the images to be similar. The use of a regular shape, in this case a circle, means that distortions are easily recognised. The introduction of dimensions in *Example 2* requires students to go beyond intuitively recognising when shapes are similar. The triangles support students' awareness of what can change and that the relationship between corresponding sides in similar

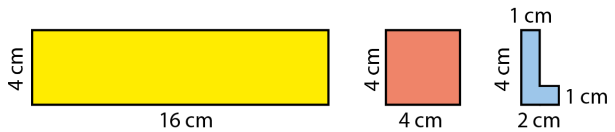
Example 2:

Tick the triangles that are similar to triangle A.



Example 3:

Emma, Feona and Georgia are asked to create a set of shapes that are similar to these.



Emma says, 'I'm going to add 5 to each length.'

Georgia says, 'I'm going to multiply each length by 0.5.'

a) *All of the methods work for one of the shapes. Which shape is it?*

b) *Which methods work for all three shapes? Would those methods work for every shape?*

shapes is multiplicative (triangles D, E and F) not additive (triangle C and G).

D Deepen students' thinking by asking more probing questions, such as:

- Find two triangles similar to triangle C.
- Find two triangles similar to triangle E with non-integer values.

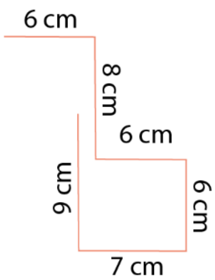
PD *Example 2* uses right-angled triangles. What range of shapes might be useful to help students refine their understanding of similarity?

Example 3 draws attention to the importance of multiplicative rather than additive enlargement.

V The square is used here as an example of a context where an additive approach will create a similar shape. Draw students' attention to this and maybe ask which other types of shape this additive approach might work for. The intention here is for students to understand this as a special case while the multiplicative approach will work for all situations.

The L-shape provides a strong representation of what happens when an additive approach is used. You might ask groups of students to 'enlarge' each shape using each method so that this can be discussed.

Here is one possible outcome of Emma's method for the L-shape:



This is clearly not similar to the original shape, even though the angles have been preserved.

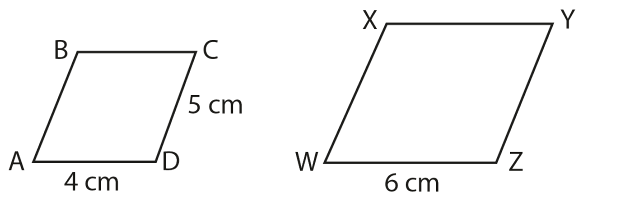
Find the scalar multiplier between two sides in similar shapes.

Example 4:

Parallelograms ABCD and WXYZ are similar.

Mary thinks YZ = 7 cm.

Do you agree? Explain your reasoning



V The 'what it's not' question in *Example 4* has been chosen to raise students' awareness that the relationships between similar shapes is multiplicative not additive and the scalar multiplier does not always have to be a whole number. In this case, because the scalar multiplier is 1.5, Mary has incorrectly chosen an additive method of adding 2 to CD rather than finding the multiplier.

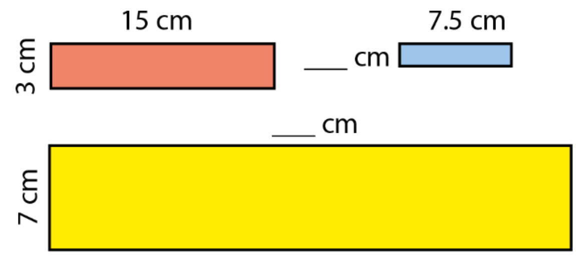
PD Do you use 'what it's not' type questions to secure students' understanding? How might we best use such questions? Verbally as discussion prompts or as written responses?

Find the functional multiplier between two sides within similar shapes

Example 5:

These three rectangles are mathematically similar.

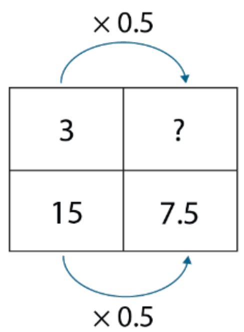
What calculations are needed to find the missing lengths?



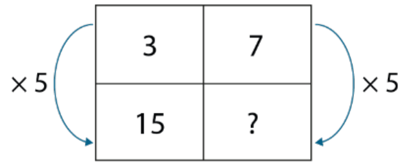
V *Example 5* is designed to raise students' awareness of the relationship between sides in the same shape (functional multiplier) and between similar shapes (scalar multiplier). It is common for students to identify the scalar multiplier more easily, even when that offers a less 'friendly' calculation than using the functional multiplier. In this case, working between the red and blue rectangle is likely to be seen as simpler than working between the red and yellow.

R Ratio tables are very powerful representations to help students notice and find both the scalar and the functional multipliers.

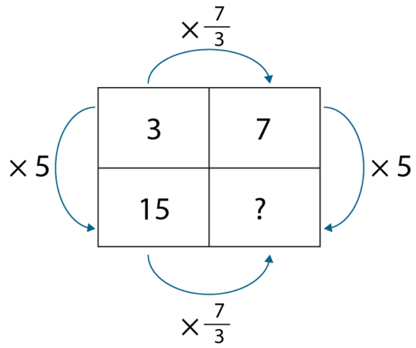
- Red to blue (scalar multiplier is shown).



- Red to yellow (functional multiplier is shown).



PD Although only one of the two possible multipliers is shown in the tables above, it is important to know that both multiplicative relationships are always present. Should both scalar and functional multipliers always be written on the ratio table as below?



Understand angles are preserved in similar shapes.

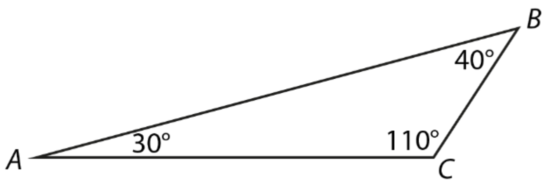
Example 6:

Steve uses a ruler and protractor to measure some triangles and their enlargements. He records the lengths and angles in a table.

Complete Steve's table.

Triangle		Scale factor	Enlarged triangle	
Dimensions (cm)	**Angles**		**Dimensions (cm)**	**Angles**
3, 3, 3	60°, 60°, 60°	2		
5, 5, 7	45°, 45°, 90°		15, 15, 21	
	60°, 60°, 60°	6	24, 24, 24	
		2	8, 12, 12	40°, 70°, 70°

V Using the triangles in *Example 2*, students can investigate if angles are preserved in similar shapes. *Example 6* is designed to support students' awareness of angles being preserved while the lengths are scaled in similar shapes.

Example 7: *The triangle is enlarged by scale factor of 2.*  *Tom thinks the value of angle ABC in the enlarged triangle will be 80°.* *Explain why Tom is not correct.*	**V**	The 'what it's not' question in *Example 7* is designed to challenge student's appreciation of angle preservation in similar shapes. In isolation, angle ABC can be doubled to 80° and still be a reasonable value for an angle in a triangle. However, if the same logic was applied to the other two angles, the angle sum would now be greater than 180°.

6.1.3.2 Use and apply Pythagoras' theorem to solve problems in a range of contexts

Common difficulties and misconceptions

Students may know the formula $a^2 + b^2 = c^2$ and be able to use it to calculate the hypotenuse in a given triangle. However, identifying where Pythagoras' theorem can be used within a problem where the triangle is not explicit, can be a challenge.

As students are introduced to trigonometric ratios and how to use these to calculate missing sides, there is a danger that this becomes the sole strategy for solving problems involving right-angled triangles and that Pythagoras' theorem might be an under-used strategy.

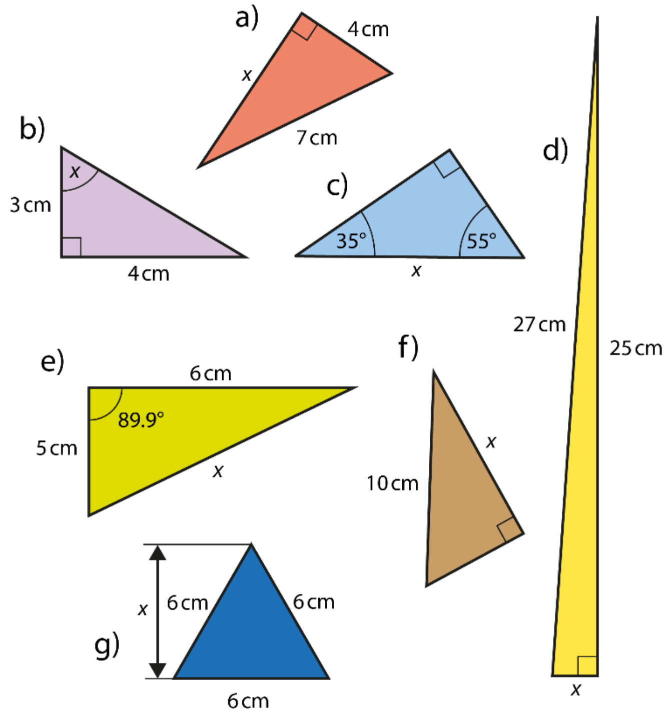
To address both these issues, it is important that students experience Pythagoras' theorem problems in many different forms, so that they are able to identify where it is an appropriate technique when solving a problem, and to deepen their understanding of the relationship that it describes.

What students need to understand

Identify when Pythagoras' theorem can be applied.

Example 1:

In which of these diagrams could Pythagoras' theorem be used to calculate x?



Not to scale

Guidance, discussion points and prompts

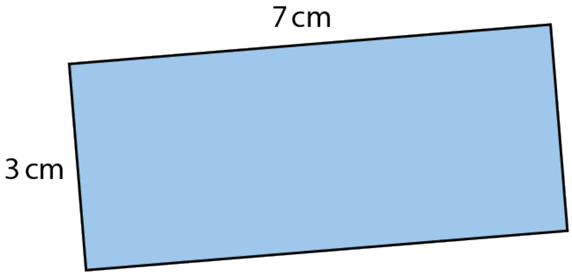
V *Example 1* offers different examples and non-examples of triangles in which Pythagoras' theorem might be used to find a missing value.

Students should identify which values of *x* can and cannot be found using Pythagoras' theorem, and should be asked to explain how they know for each example.

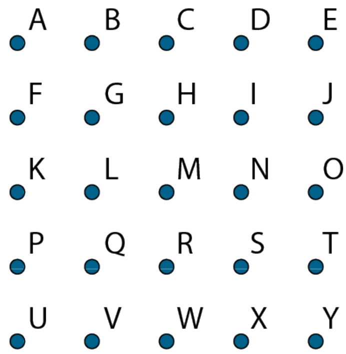
Solve problems using Pythagoras' theorem to calculate lengths.

Example 2:

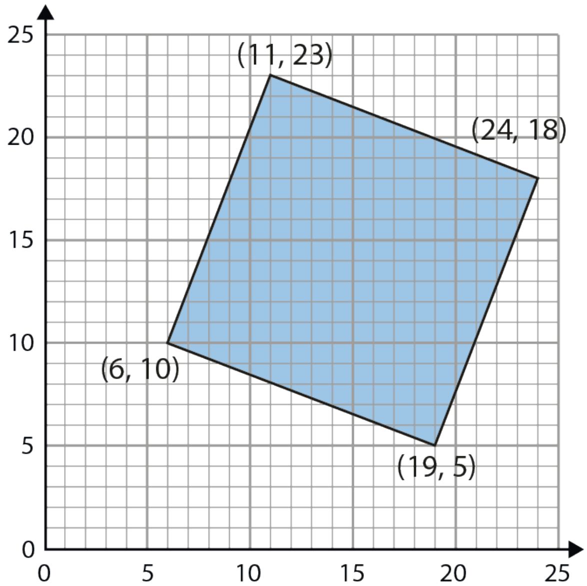
a) *Can a straight line that is 8 cm long be drawn inside this rectangle?*



b) *This is a 1 cm grid. Which two points should be joined to make a straight line that is exactly 5 cm long?*



c) *Find the perimeter of this square.*



D Each of the problems in *Example 2* uses Pythagoras' theorem to find a missing length.

Changing the context in which a concept is practised can help students identify the key features of that concept. In this instance, students are asked to calculate a length in problems where a right angle is evident, but no other angles are offered or required.

A key understanding here is that Pythagoras' theorem can only be used to find a missing length where the triangle contains a right angle. Identifying the right angle (where it is not obvious) and/or the triangle is a key part of solving these problems. Students should be encouraged to reflect on the similarities and differences across these questions. A prompt, such as, '*How did you know that you could use Pythagoras' theorem to solve this problem?*', may be helpful.

Alternatively, the problems may be set as a single exercise in which students' first task is to identify the right angle and the triangle in each case (using a prompt, such as, *'Each of these problems involves Pythagoras' theorem. Identify the triangle and relevant lengths in each one.'*) before then revisiting the tasks to find a numerical solution.

You may also like to combine this set of questions with other triangle questions and ask students to identify which they could use Pythagoras' theorem to solve. This may, again, draw students' attention to the key features of Pythagoras' theorem.

PD What teaching techniques do you use in your classroom to draw out and make explicit the strategies that students use to identify what mathematical knowledge is appropriate in solving a problem?

How can you support your students to identify them (and other problems they have not yet met) as examples of Pythagoras' theorem?

d) *Brighton is 48 miles south of London.*

Cardiff is 130 miles west of London.

(i) *What is the direct distance between Brighton and Cardiff?*

(ii) *Can you identify three different cities that might be able to replace London, Brighton and Cardiff in this question?*

What is it about them that makes them suitable?

e) *When using a ladder, it is recommended to use the four-to-one rule: for every four feet the ladder goes up the wall, move the base away from the wall by one foot.*

Using this rule, how high up a wall should a 24-foot ladder reach?

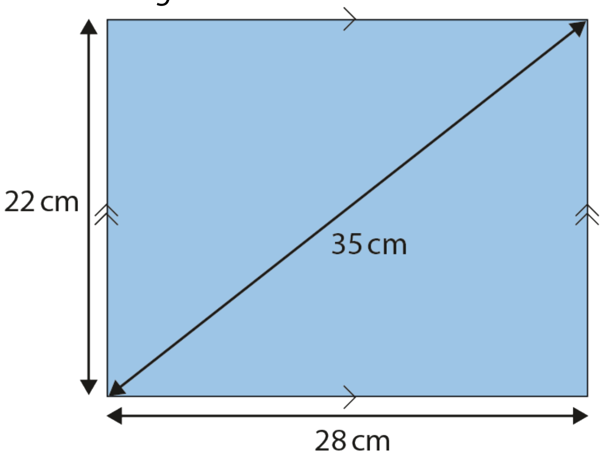
Consider other contexts in which Pythagoras' theorem might be applied. Can you add to this list?

- Find the height/area of an isosceles/equilateral triangle.
- Find the area of an isosceles trapezium given lengths of parallel sides and sloping height.
- Simple bearings questions, such as: a boat travels north 50 km, then due east 70 km, how far away is it from port?
- Find the length of a line segment given the coordinates of the end points.
- Find the surface area of a right-angled triangular prism with the hypotenuse not given (so that students need to find the area of the sloping rectangular face).
- Which TV to buy? What does 42-inch screen mean?

Solve problems using Pythagoras' theorem to make statements about angles.

Example 3:

a) *The shape below shows a parallelogram. Is it also a rectangle?*

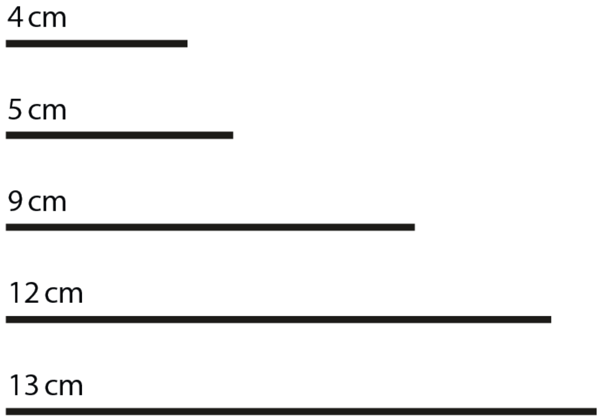


Each of the problems in *Example 3* asks students to make a decision about a shape. The use of Pythagoras' theorem as a strategy is perhaps less obvious than when the length of a side is being found, but the focus here is on the angles within the triangle, and the use of the right angle.

In parts a) and b), students use Pythagoras' theorem to identify whether or not an angle is 90°. In parts b) and c), students can be asked to visualise the impact of the changes before going on to calculate to test their predictions.

Further prompts might be used to encourage students to further explore a situation.

In part a), (which is not currently a rectangle) you might ask students to consider what the diagonal *b)*



(i) Using three of these rods, is it possible to make a triangle that contains an obtuse angle? If so, which three rods? Is there more than one set?

(ii) Using three of these rods, is it possible to make a triangle that contains a right angle? If so, which three rods? Is there more than one set?

(iii) Using three of these rods, is it possible to make a triangle that contains all acute angles? If so, which three rods? Is there more than one set?

c) Darcy draws a right-angled triangle.

Emil says, 'I'm going to draw a bigger right-angled triangle by making each side of your triangle 10 cm longer.'

Will Emil's triangle be a right-angled triangle? Will it have an obtuse angle?

should be if the height and width are measured correctly, or what the height should be if the base and diagonal are measured correctly, etc.

You might also ask students to visualise and sketch what the shape looks like given its current dimensions. How far is it from being a rectangle?

In part b), students might consider the minimum number of rods needed to allow all three types of triangle to be made – can they find a set of four rods which can be combined in different ways to make three different types of triangle?

In part c), students can use what they know about similar shapes to see that adding the same amount to each length cannot create a similar shape, but might then be encouraged to use similarity to find Pythagorean triples.

In each of these problems, Pythagoras' theorem is being used to give a snapshot of a situation with a right angle, which can then be explored as one or more of the dimensions changes.

Weblinks

1 NCETM primary mastery professional development materials
https://www.ncetm.org.uk/resources/50639

2 Standards & Testing Agency past mathematics papers
https://www.gov.uk/government/collections/national-curriculum-assessments-practice-materials#key-stage-2-past-papers