MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

Multiplication and Division

2.3 Times tables: groups of 2 and commutativity (part 1)

Teacher guide | Year 2

Teaching point 1:

For equally grouped objects, the number of groups is a factor, the group size is a factor, and the overall number of objects is the product; this can be represented with a multiplication equation. Counting in multiples of two can be used to find the product when the group size is two.

Teaching point 2:

Counting in multiples of two can be represented by the two times table. Adjacent multiples of two have a difference of two. Facts from the two times table can be used to solve problems about groups of two.

Teaching point 3:

Factor pairs can be written in either order, with the product remaining the same. (commutativity)

Overview of learning

In this segment children will:

- practise counting in multiples of two
- connect enumerating a number of objects by skip counting in twos (for example, three pairs of shoes is two, four, *six* shoes) with representing the grouped objects with a multiplication equation ($3 \times 2 = 6$)
- use the language of factors and product to describe multiplication equations
- build up the two times table by combining their understanding of writing multiplication equations and their knowledge of skip counting in twos
- explore patterns in the two times table
- explore the fact that adjacent multiples of two have a difference of two, and represent this with mixed operation equations (e.g. $4 \times 2 = 3 \times 2 + 2$ and $3 \times 2 = 4 \times 2 - 2$)
- practise writing multiplication equations to represent groups of two, understanding that the factors can be written in either order, with the product remaining the same.

This segment draws together the learning from segments *2.1 Counting, unitising and coins* and *2.2 Structures: multiplication representing equal groups*. In segment *2.1* children learnt to enumerate objects arranged in groups of two, for example three pairs of shoes are counted as '*two, four, six*' to find that the total number of shoes is six. In segment *2.2*, children learnt to write multiplication expressions to represent equal groups, for example three pairs of shoes can be represented with the expression 3×2. In *Teaching point 1* of this segment, these concepts are connected as children learn to write multiplication *equations* for the first time (as opposed to multiplication *expressions*); for example the context of the three pairs of shoes can now be represented by the equation $3 \times 2 = 6$ (and $6 = 3 \times 2$). Children will learn that the numerals representing the number of groups and the group size in a multiplication equation are both called *'factors'* and that the numeral representing the total number of objects is called the *'product'*:

- factor × factor = product
- product = factor × factor

As discussed in segment *2.2*, *Overview of learning*, up until segment *2.12 Division with remainders*, the language of *'multiplied by'* is not used, as this implies that the multiplicand (the group size) is written first followed by the multiplier (the number of groups); children instead use the language '*groups of*', '*times*' (not '*times* by') and '*___ twos*' (for example, *'four twos'*). In *Teaching points 1* and *2* of this segment, children will continue to write the factors in the order *number of groups* × *group size*. The generalisation '*factor* × *factor* = *product*' is used, since in *Teaching point 3*, children will learn that the factors can be written in either order; for example, three groups of two can be represented by both equations:

- $3 \times 2 = 6$
- $2 \times 3 = 6$

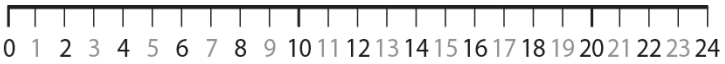
The language of '*three groups of two*' is contrasted with *'two, three times'* to connect the equations to the structure of the context. From this point on, times tables will always be written out 'both ways'. Note that the commutativity of the factors discussed in this segment remains linked to contexts with groups of two (one interpretation, two equations). In segment *2.5 Commutativity (part 2), doubling and halving*, commutativity will be explored in the sense that, for example, 3×2 can represent *both* three groups of two *and* two groups of three (one equation, two interpretations). At this early stage, however, it is important to keep a strong link between group size and the times table being learnt.

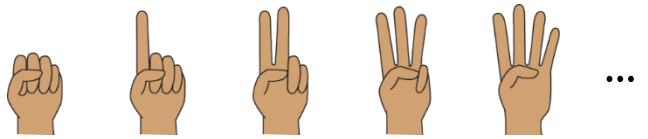
An explanation of the structure of these materials, with guidance on how teachers can use them, is contained in this NCETM podcast: www.ncetm.org.uk/primarympdpodcast. The main message in the podcast is that the materials are principally for professional development purposes. They demonstrate how understanding of concepts can be built through small coherent steps and the application of mathematical representations. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

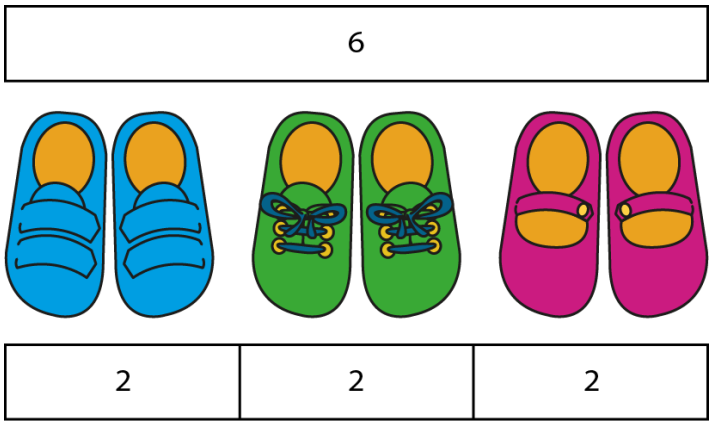
Teaching point 1:

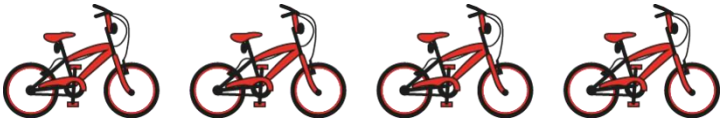
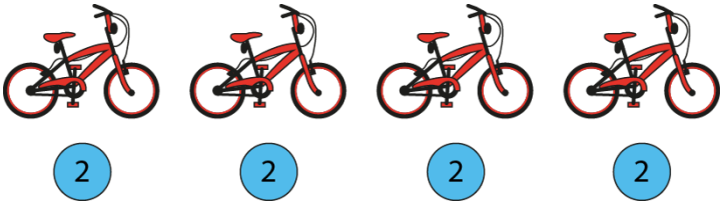
For equally grouped objects, the number of groups is a factor, the group size is a factor, and the overall number of objects is the product; this can be represented with a multiplication equation. Counting in multiples of two can be used to find the product when the group size is two.

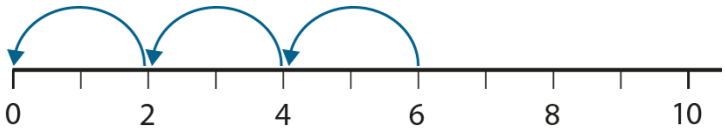
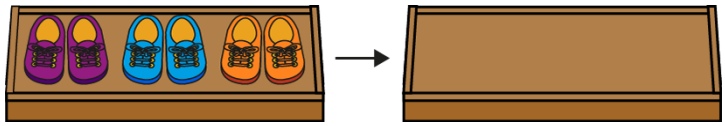
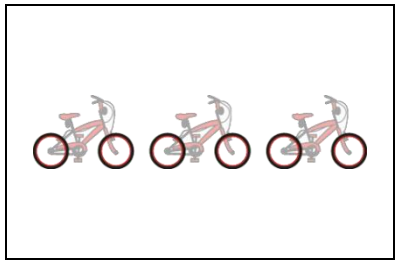
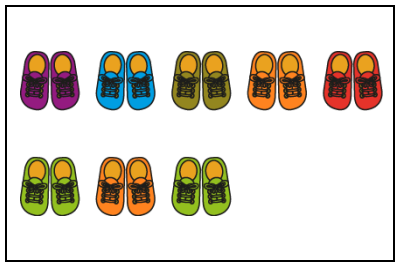
Steps in learning

	Guidance	Representations
1:1	Begin by briefly reviewing counting in multiples of two, which was introduced in segment *2.1 Counting, unitising and coins*. At this point, you can extend the counting up to twelve twos. Use familiar representations, including those from segment *2.1*, including: • a number line • the Gattegno chart (have children tap the chart as they count: one tap for the single-digit numbers; two taps for the two-digit numbers, for example, tap '10' then '2' on the count of 12) • objects that come in pairs (for example pairs of shoes, eyes, ears and so on; 2 p pre-money tokens) • using skip counting to find the value of a number of 2 p coins. As in segment *2.1*, count in two ways: • *'Zero groups of two, one group of two, two groups of two, three groups of two…'* • *'Zero, two, four, six…'* You can begin to shorten the former to: *'Zero twos, one two, two twos, three twos…'* Skip counting in twos beyond 20 may be new to children, but as they are procedurally familiar with both the pattern and the representations used, they should be able to do this and discuss how they know what comes next. Practise counting both forwards and backwards in twos from different	Skip counting in twos – number line:  Gattegno chart: 1000 2000 3000 4000 5000 6000 7000 8000 9000 100 200 300 400 500 600 700 800 900 10 20 30 40 50 60 70 80 90 1 2 3 4 5 6 7 8 9 Skip counting in twos – objects in pairs: 

	multiples of two. During forwards counting with number names (*'Zero, two, four, six…'*) encourage children to keep a tally with their fingers of the number of groups of two counted; note that the image opposite illustrates, for teachers only, how each finger represents one *group of two* (not one) – here children will be unitising in groups of two using their fingers to 'silently' note the number of groups of two as they count out the number names. Follow a similar process for backward counting.	Skip counting in twos – two-pence coins:  Skip counting in twos – keeping a 'silent' tally of the number of groups of two using fingers:  *'Zero, two, four, six, eight …'*
1:2	Now connect enumerating the number of objects using skip counting (for example, three pairs of shoes is six shoes) with representing the grouped objects with a multiplication expression (e.g. 3×2), as shown opposite. Notice how the same language has been used opposite as in segments *2.1 Counting, unitising and coins* and *2.2 Structures: multiplication representing equal groups*; now we are bringing both together. Model how the multiplication *expression* can be turned into an *equation* showing the product, building up a bar model alongside the pictorial representation to represent the structure. Show the multiplication equation written both ways – with the product at the end ($3 \times 2 = 6$) and with the product at the start ($6 = 3 \times 2$). Referring to the final equation(s), ask children to describe what each number represents:	Enumerating the objects by skip counting in twos:  • *'How many shoes are there? Count in groups of two.'* • *'Two, four, six. There are six shoes.'* 6 

	• *'What does the "3" represent?'* *'The "3" represents the number of pairs/groups of shoes.'* • *'What does the "2" represent?'* • *'The "2" represents the number of shoes in a pair/group.'* • *'What does the "6" represent?'* • *'The "6" represents how many shoes there are altogether.'* Note, when describing a multiplication fact such as '$3 \times 2 = 6$' use the language '*three times two is equal to six*'. Avoid saying '*times by*' or '*multiplied by*'. For more on this, see segment *2.2, Overview of learning*.	Describing and representing the grouped objects: • *'How many groups of shoes are there?'* • *'How many shoes are there in each group?'* • *'There are three groups of shoes. There are two shoes in each group.'* • *'There are three groups of two.'*  Combining the information: $3 \times 2 = 6$ $\quad$ $6 = 3 \times 2$ • *'There are three groups of two; there are six altogether.'*
1:3	Continuing with the example in step *1:2*, introduce the word '*product*'. Referring to the bar model and multiplication equation, ask children: • *'What does the "3" represent?'* • *'What does the "2" represent?'* • *'What does this "6" represent?'* Then explain that we can say: • *'Six is the product of two and three.'* • *'The product of two and three is six.'* Ensure that the children understand the word 'product' to mean how many there are altogether. Then introduce the word '*factor*': • *'Three is a factor.'* • *'Two is a factor.'* Then write out the structure: • factor × factor = product • product = factor × factor	 

	Use the generalised statements: • ***'Factor times factor is equal to the product.'*** • ***'The product is equal to factor times factor.'***	
1:4	Work through other examples with a group size of two, now embedding the '*factor–factor–product*' language. You can begin to use counters marked with the value '2' as a generalised representation of various contexts. Explain to children that each counter represents a group of two in the context (ensuring that children don't see them as additional twos; for example each counter opposite represents two wheels, the combined representations do not show eight twos). As with the bar model, this generalised representation will begin to emphasise the similarity between different contexts. It is worth noting that the counters don't *need* to have anything written on them for us to attribute a value to them; we can simply decide that, at the moment, a counter represents two; later we might decide that a counter represents five, or 0.5 etc. However, it is useful to include the written numeral at this stage; some children may find it too challenging to unitise with blank counters whilst focusing on the structure being explored. In the example opposite, we begin by defining the group size and number of groups, before looking to find the product (the opposite order to that in step *1:3*). For now, children should continue to use skip counting to find the product. The two times table and the importance of memorising multiplication facts are discussed in the next teaching point.	• *'How many bicycles are there?'* • *'How many wheels does each bicycle have?'*  • *'There are four bicycles.'* • *'Each bicycle has two wheels.'* • *'There are four groups of two.'* 4×2 • *'How many wheels are there altogether?'*  • *'Two, four, six, eight.'* • *'There are eight wheels.'* $4 \times 2 = 8$ $8 = 4 \times 2$ • *'Four is a factor.'* • *'Two is a factor.'* • *'The product of four and two is eight.'* • *'Eight is the product of four and two.'*

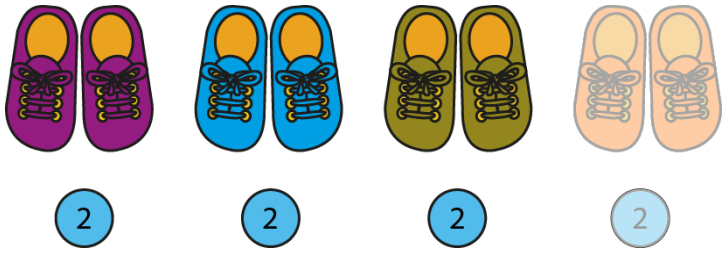
1:5	Now explore 0×2. Use skip counting backwards in groups of two, supported by the number line, starting with, for example, three groups of two. Write the multiplication equation in each case. You can use a contextual representation alongside the number line, such as the pairs of shoes shown opposite. Remove a pair of shoes, one at a time, until there are zero pairs of shoes and, therefore, zero shoes.	Backward counting to reach zero twos:  • *'Six, four, two, zero.'* • *'Three twos, two twos, one two, zero twos.'*  • $3 \times 2 = 6$ • $2 \times 2 = 4$ • $1 \times 2 = 2$ • $0 \times 2 = 0$
1:6	Provide children with practice as shown opposite, and problems such as *'I have five packets of biscuits. Each packet contains two biscuits. That's five groups of two. I have ten biscuits altogether. Draw a picture and write an equation to represent this.'* For problems without pictorial contexts, encourage children to use or draw counters with a value of two to represent the descriptions/equations.	• *'Match each equation with the correct picture.'*  $8 \times 2 = 16$  $5 \times 2 = 10$  $3 \times 2 = 6$

		• *'Iniko started drawing some pairs of mittens. Complete his drawing to match the equation.'* 12 = 6 × 2  • *'Which of these <u>doesn't</u> represent groups of two?'* $0 \times 2 = 0$ $4 \times 3 = 12$ $4 \times 2 = 8$ $3 \times 2 = 6$

Teaching point 2:

Counting in multiples of two can be represented by the two times table. Adjacent multiples of two have a difference of two. Facts from the two times table can be used to solve problems about groups of two.

Steps in learning

	Guidance	Representations
2:1	Now, using a familiar context alongside the two-value counters, work systematically to build up the two times table (to 12 twos) as you count in multiples of two, beginning with zero twos. Use a ratio table to record the number of groups and the product. As you complete the ratio table, also write the multiplication equations, recording the number of groups as the first factor, and the group size (in this case '2') as the second factor. At each stage, encourage children to relate the equation and table to the context, for example: • *'There are three pairs/groups of two shoes.'* • *'There are six shoes altogether.'* • *'The product of three and two is six.'* Then add another pair of shoes, and work with children to complete the next column of the table, using their knowledge of what comes next in the counting sequence when skip counting in twos.	Building up the two times table: 

	Number of *pairs* of shoes	Number of shoes
$0 \times 2 = 0$	0	0
$1 \times 2 = 2$	1	2
$2 \times 2 = 4$	2	4
$3 \times 2 = 6$	3	6
$4 \times 2 = \mathbf{8}$	4	**8**

2:2

Once the table and full set of equations are complete, ask children questions, encouraging them to use the table/equations for support, for example:

- *'If I have six* pairs *of shoes, how many shoes do I have altogether?'*
- *'If I have four shoes, how many pairs do I have?'*

Complete two times table and ratio chart:

	Number of *pairs* of shoes	Number of shoes
$0 \times 2 = 0$	0	0
$1 \times 2 = 2$	1	2
$2 \times 2 = 4$	2	4
$3 \times 2 = 6$	3	6
$4 \times 2 = 8$	4	8
$5 \times 2 = 10$	5	10
$6 \times 2 = 12$	6	12
$7 \times 2 = 14$	7	14
$8 \times 2 = 16$	8	16
$9 \times 2 = 18$	9	18
$10 \times 2 = 20$	10	20
$11 \times 2 = 22$	11	22
$12 \times 2 = 24$	12	24

2:3

Now present a multiplication problem such as *'There are four nests. Each nest has two eggs. How many eggs are there altogether?'*

Ask children to represent the problem with two-value counters, or with a drawing, and write a multiplication expression (4×2). Then ask them to suggest how we can work out the answer. Suggestions may include:

- counting the eggs one by one
- counting the eggs in twos
- using the multiplication chart
- remembering that four times two is equal to eight.

Explain that if we learn our multiplication facts, we can solve problems like this more efficiently.

2:4

Now practise chanting the two times table, as a class, using a variety of representations, including:

- a number line
- the Gattegno chart
- concrete representations
- pictorial representations.

Use the written times table for support. As children gain confidence, include contexts where the cardinality isn't apparent, for example, 2 p coins.

Regular practice should be undertaken, including outside the main maths lesson (perhaps using classroom routines, such as counting pairs of children), until children are fluent.

Use the following shortened language to support memorisation: *'One two is two, two twos are four, three twos are six…'*

Some children may enjoy being challenged to say how many socks there would be if there were one hundred pairs, one thousand pairs or one million pairs. They can answer these questions just through attention to language patterns.

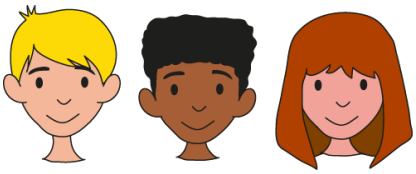
Chanting the two times table – pictorial context with cardinality visible:

 $1 \times 2 = 2$	*'One two is two.'*
 $2 \times 2 = 4$	*'Two twos are four.'*
 $3 \times 2 = 6$	*'Three twos are six.'*
⋮	⋮

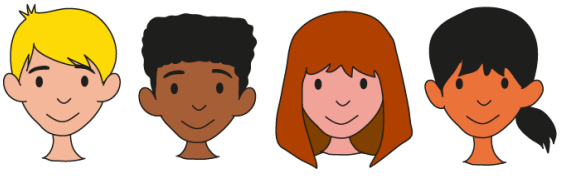
2:5

Provide children with some practice, as shown opposite and below. At this stage, they can recite the two times table up to the required number of twos in order to find the answers, or use the multiplication chart for reference. Plenty of practice will be needed over an extended period until children are fluent in the isolated multiplication facts (for example just *knowing* that seven twos is equal to fourteen, rather than having to recite the times table up to seven twos). Some children may already be making the link with their known doubles (*Spine 1: Number, Addition and Subtraction*, segment *1.7*); segment *2.5*

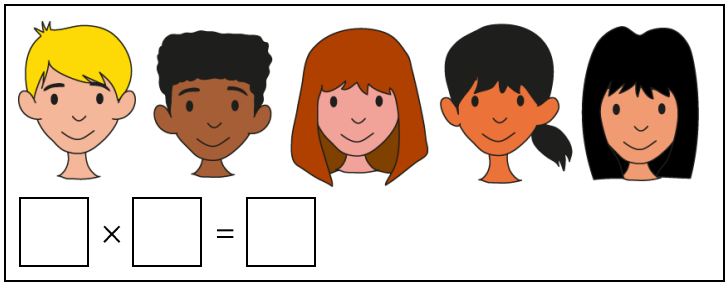
'For each picture, complete the equation to show how many eyes the children have altogether.'



$3 \times 2 =$ ☐



☐ $\times 2 =$ ☐

	Commutativity (part 2), doubling and halving, explores this in more detail. Example oral/word problems: • *'What is the product of four and two?'* • *'I have eleven 2 p coins. How much is this altogether?'* • *'If there are five pairs of children, how many children are there altogether?'* • *'If ten children line up in twos, how many twos will there be?'* • *'If the product is twelve and one factor is two, what is the other factor?'* Encourage children to write a multiplication equation for each problem, rather than simply writing the product.	
2:6	Now explore patterns in the two times table. Show the multiplication chart and ask children what patterns they can see. Draw out the following: • Working down the list, the first factor in each equation increases by one each time. • The second factor in each equation is two. • The products are all even numbers. • Working down the list, the product increases by two each time. Ask children to use their previous learning on even numbers (*Spine 1: Number, Addition and Subtraction*, segment *1.7*) to explain the connection between the last two points above, perhaps using the number line for support: • *'Adding two to an even number gives the next even number.'* • *'Subtracting two from an even number gives the previous even number.'* • *'Consecutive even numbers have a difference of two.'* • *'So, adjacent multiples of two have a difference of two.'*	

2:7

Focus in on the fact that adjacent multiples of two have a difference of two, and that this knowledge can be used to find the next or previous multiple of two from a given multiple. Present a ratio chart with some missing numbers, and work with children to use the adjacent multiples rule to fill in the missing numbers. You can use a number line with a bunny/frog beginning at the known fact, then jumping two to get to the missing fact.

Also, begin to expose children to the idea of arrays, adding or removing (revealing or hiding) a row to move from the known multiple to the next/previous multiple. You can use a portion of a 12×12 chart as shown opposite.

Throughout, write the corresponding equation to represent the operations, for example:

- $3 \times 2 = 2 \times 2 + 2$
- $7 \times 2 = 8 \times 2 - 2$

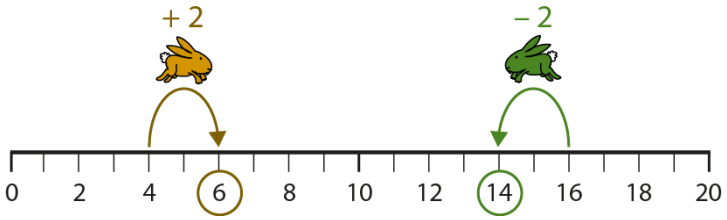
Note that at this stage we are only looking at adjacent multiples. The distributive law will be explored fully in segment *2.9 Commutative and distributive laws of multiplication* (e.g. $7 \times 2 = 3 \times 2 + 4 \times 2$).

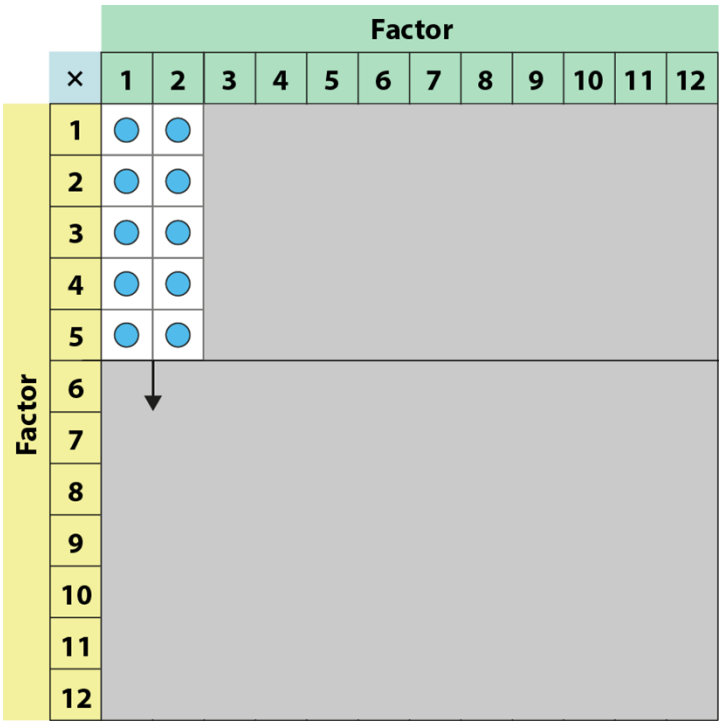
Finding adjacent multiples – ratio chart and number line:

	× 2
0	0
1	2
2	4
3	
4	8
5	10
6	12
7	
8	16
9	18
10	20
11	22
12	24

$\downarrow + 2$ $\quad 3 \times 2 = 2 \times 2 + 2$

$\uparrow - 2$ $\quad 7 \times 2 = 8 \times 2 - 2$



		Finding adjacent multiples – array chart:  $6 \times 2 = 5 \times 2 + 2$

2:8	Provide children with varied practice, applying what they have learnt throughout this teaching point, including:
	• ratio tables with missing numbers (ask children to explain how they know what the missing number is using their knowledge of adjacent multiples of two having a difference of two)
	• sequences of missing-number problems
	• true/false style questions
	• real-life problems, including measures contexts, for example:
	  • *'Ignacio looks in his sock drawer and sees four pairs of red socks.'*
	  • *'How many socks is this altogether?'*
	  • *'Ignacio sees that there is also a pair of blue socks. How many red and blue socks are there altogether?'*
		Missing-number problems:
		• *'Fill in the missing numbers.'*
		
		× 2
		0 __
		__ 2
		2 4
		__ 6
		4 __
		5 __
		__ 12
		7 __
		__ 16
		__ 18
		10 20
		11 22
		12 __
		
		$10 \times 2 = 20$
		$9 \times 2 =$ ☐
		$8 \times 2 =$ ☐
		$7 \times$ ☐ $= 14$
		$6 \times$ ☐ $= 12$
		$5 \times$ ☐ $= 10$
		☐ $\times 2 = 8$
		☐ $\times 2 = 6$
		☐ $\times 2 = 4$

- *Dòng nǎo jīn: 'Ignacio then finds one odd sock. How many socks does he now have altogether?'*
- *'Jason has seven two-litre bottles of water. '*
  - *'How many litres is this altogether?'*
  - *'Jason gives one of the two-litre bottles of water away. How many litres of water does he have now?'*

Dòng nǎo jīn: *'Five children go out wearing gloves. Then one child loses a glove. How many gloves are there now?'*

$5 \times 2 = 4 \times 2 + \square$ $\quad$ $8 \times 2 = \square \times 2 + 2$

$5 \times 2 - \square = 4 \times 2$ $\quad$ $8 \times 2 - 2 = \square \times 2$

- *'Fill in the missing symbols.'*

$7 \times 2 \bigcirc 6 \times 2 - 2$

$7 \times 2 \bigcirc 6 \times 2 + 2$

$7 \times 2 \bigcirc 8 \times 2 + 2$

$7 \times 2 \bigcirc 8 \times 2 - 2$

True or false questions:

- *'Is this sentence true or false?'*

If 9 is a factor and 2 is the product, 18 is a factor.

- *'These sets of numbers are products in the two times table.'*

	These are products in the two times table. True (✓) or false (✗)?
4, 6, 7, 12, 16	
18, 10, 0, 6, 14	
16, 4, 20, 2, 8	
32, 33, 43, 55, 64	
58, 60, 62, 64, 68	

Teaching point 3:

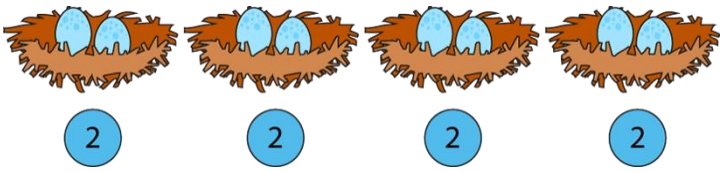
Factor pairs can be written in either order, with the product remaining the same. (commutativity)

Steps in learning

	Guidance	Representations
3:1	In this teaching point children will explore how the factors in a multiplication equation (representing groups of two) can be written in either order, with the product remaining the same (we will refer to this as the *'one interpretation, two equations'* concept of commutativity), i.e.: • no. of groups × group size = product • group size × no. of groups = product The 'one equation, two interpretation' concept of commutativity will be covered in *2.5 Commutativity (part 2), doubling and halving* (for more, see *Overview of learning* above). Begin by looking at the two times table multiplication chart from *Teaching point 2*. Ask children to explain what the numbers represent: • *'The first factor represents the number of groups.'* • *'The second factor (always "2") represents the group size.'* then recite the two times table together using the language: • *'One group of two is equal to two, two groups of two is equal to four, three groups of two is equal to six…'* • *'One times two is equal to two, two times two is equal to four, two times three is equal to six…'* Now show how you can record the two times table with the factor '2' (the group size) written first. Record this alongside the existing times table to highlight the equivalence between the	$0 \times 2 = 0$ | $2 \times 0 = 0$ $1 \times 2 = 2$ | $2 \times 1 = 2$ $2 \times 2 = 4$ | $2 \times 2 = 4$ $3 \times 2 = 6$ | $2 \times 3 = 6$ $4 \times 2 = 8$ | $2 \times 4 = 8$ $5 \times 2 = 10$ | $2 \times 5 = 10$ $6 \times 2 = 12$ | $2 \times 6 = 12$ $7 \times 2 = 14$ | $2 \times 7 = 14$ $8 \times 2 = 16$ | $2 \times 8 = 16$ $9 \times 2 = 18$ | $2 \times 9 = 18$ $10 \times 2 = 20$ | $2 \times 10 = 20$ $11 \times 2 = 22$ | $2 \times 11 = 22$ $12 \times 2 = 24$ | $2 \times 12 = 24$

	pairs of equations. Recite the two times table again, now using the language • *'Two, one time is equal to two, two, two times is equal to four, two, three times is equal to six…'* • *'Two times one is equal to two, two times two is equal to four, two times three is equal to six…'* For now, avoid saying *'two once'* and *'two twice'*, instead retaining the number names 'one' and 'two' for the number of groups.	

3:2	Now draw attention to a particular pair of equations and compare them, for example:
	$5 \times 2 = 10$
	$2 \times 5 = 10$
	• *'What's the same?'*
	  • *'In both equations "2" and "5" are factors and "10" is the product.'*
	• *'What's different?'*
	  • *'The factors are written in a different order.'*	

3:3	Now look at a contextual example. Present groups of two, as shown opposite, and ask children to describe the context and write equations to represent it. Ask children to explain what each number represents.	 $2 + 2 + 2 + 2 = 8$ • *'There are four groups of two eggs. There are eight eggs altogether.'* $4 \times 2 = 8$ • *'There are two eggs, four times. There are eight eggs altogether.'* $2 \times 4 = 8$

3:4	Provide children with some practice, including sequences of missing-number problems and writing pairs of equations to describe groups of two. Encourage children to continue to use the language of, for example, *'four groups of two'* and *'two, four times'* for support.	Missing-number problems: *'Fill in the missing numbers.'* $0 \times 2 = 2 \times 0$ $1 \times 2 = 2 \times \square$ $2 \times 2 = 2 \times \square$ $3 \times 2 = \square \times 3$ $4 \times 2 = \square \times 4$ $5 \times \square = 2 \times 5$ $\square \times 2 = \square \times 6$ $2 \times 12 = 12 \times 2$ $2 \times 11 = 11 \times \square$ $2 \times 10 = \square \times 2$ $2 \times \square = 9 \times 2$ $\square \times 8 = 8 \times 2 \times 2$ $2 \times \square = 7 \times \square$ Writing pairs of equations to describe groups of two: *'Write two different multiplication equations to represent each picture.'*  $4 \times 2 = \square$ $2 \times \square = 8$  $\square \times \square = \square$ $\square \times \square = \square$