Primary & Early Years Magazine 100

National Centre for Excellence in the Teaching of Mathematics

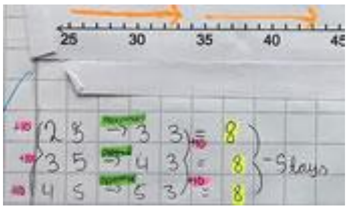
Welcome to Issue 100! In celebration of reaching this century of Primary Magazines, we're asking Can You Make 100? This task from NRICH is suitable for developing problem solving and practising fluency with pupils in KS2 and KS3. And on that theme, this is a special edition of our Primary Magazine, examining continuity of approach from primary to secondary school, a transition that has long been identified as a time when pupils can experience a dip in both achievement and engagement in maths.

By focusing on two areas of maths in particular - fractions and subtraction - our two articles suggest more generally, how the use of consistent models, images and emphasis on deep understanding can be maintained across the phases. The intention is that the maths at KS3 feels familiar to pupils making the transition, so that they can build on what they already know, rather than having to learn it all again 'because we do it like **this** at secondary'. Also, building strong models of concepts in primary school will support much more complex maths, as pupils progress through secondary school and beyond.

Continuity through the Y5-8 transition is a theme being developed by teachers in one of the Maths Hubs' Network Collaborative Projects. For more information or to get involved, contact your local Maths Hub.

Don't forget all previous issues are available in the Archive.

This issue's featured articles



Adding meaning to subtracting
Robert Wilne is one of the Work Group Leads for the 'Improving Continuity Across the Y5-8 Transition' Network Collaborative Project. In this article, he argues that strong and explicit models of subtraction should be continuous throughout the school phases. He shows that primary children can be taught to represent and think about subtraction in a way that will be just as applicable to algebraic contexts in KS3 and beyond, as it is to numerical problems in KS1 and 2.

$$\frac{\square}{15} + \frac{\square}{10} =$$

Continuity and development in the teaching of fractions across KS2 and 3
This article looks at the fundamental concepts in the area of fractions that the curriculum requires children to engage with in primary school, and how these develop in KS3. For each of these, there is an example question intended to reveal the depth of children's understanding of fractions, and their ability to reason their way through unfamiliar problems.

And here are some other things for your attention:

- Due to significant new funding in support of our work, the NCETM and Maths Hubs are expanding, with a number of new posts being advertised. See our recruitment page for details
- Our most recent podcast features a conversation with Jonathan Leeming, a Primary Mastery Specialist from Lancaster, who wrote about Teacher Research Groups in the last issue of the magazine
- The EEF has published a report, Improving Mathematics in Key Stages Two and Three, with eight detailed recommendations for good practice. In this blog, one of the lead researchers explains how these recommendations align with a teaching for mastery approach
- Our professional development materials for primary teachers using a teaching for mastery approach are now complete for the strand Y1: Number, Addition and Subtraction
- We're launching our first Facebook group! Aimed at KS1 teachers, we are promoting lively discussion

and sharing of observations, experiences, suggestions, questions and thoughts about maths in Y1 and Y2 classrooms. You can apply to join now – we'll be opening it up for discussion on 1 February

- Last November, Ofsted published Bold beginnings, an analysis of curriculum provision for the Reception year in a sample of good and outstanding primary schools
- NSPCC Number Day takes place on 2 February: register now to involve your pupils in a national fundraising day that raises the profile of mathematics.

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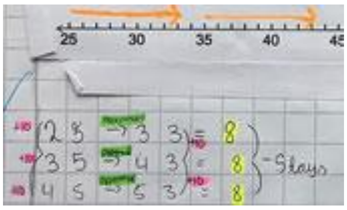


National Centre for Excellence in the Teaching of Mathematics

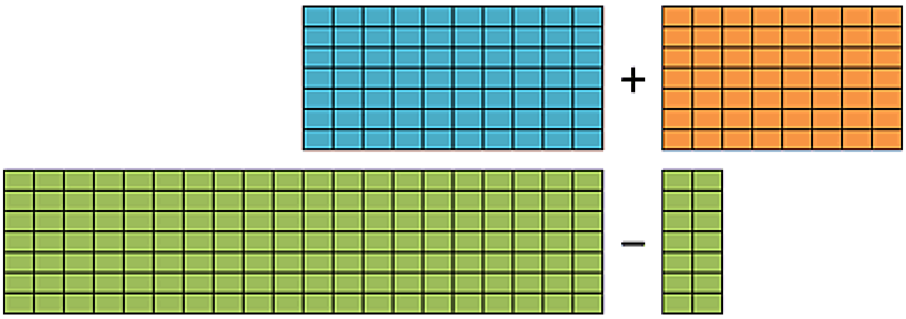
Adding meaning to subtracting

Robert Wilne is Deputy Director (Maths) at the Atlas TSA family of seven primary and secondary schools in SE London, where he is leading the development of their 3-19 maths curriculum. He's also the London Thames Maths Hub Work Group Lead for the 'Improving Continuity Across the Y5-8 Transition' Network Collaborative Project.

We often say that we want our pupils to have more than one strategy for solving a problem. Sometimes the different procedures they suggest are not conceptually significantly different, for example working out 18×7 as

- either $10 \times 7 + 8 \times 7$
- or $20 \times 7 - 2 \times 7$

is in both cases an application of the Distributive Law of Multiplication over Addition or Subtraction:



Sometimes, however, they are different. Consider this SATS question:

19 Amina posts three large letters.

The postage costs the same for each letter.

She pays with a £20 note.

Her change is £14.96

What is the cost of posting one letter?

The first step is to calculate "£20 subtract £14.96".

Some pupils will apply the column subtraction algorithm and get the answer "£5.04", assuming they navigate (twice!) the difficulty of exchanging between adjacent columns.

But others will reason

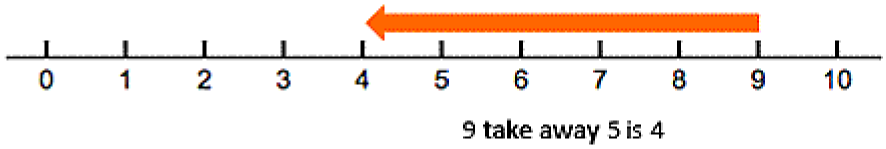
"£14.96 add on 4p and then add on £5 is £20, so the difference between £14.96 and £20 is £5.04".

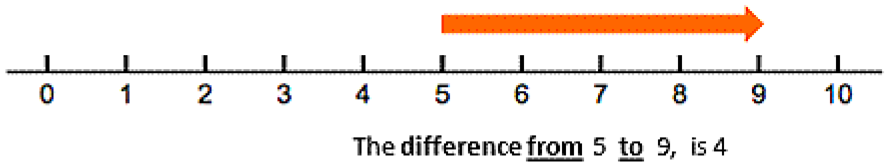
Here, the two different procedures are not conceptually the same, and the distinction between them is important and powerful.

A subtraction such as:

"9 – 5 = 4"

is an abstract representation of the two very different concrete processes of 'take away' and 'difference between'. The **same** string of symbols represents two very **different** pictures:

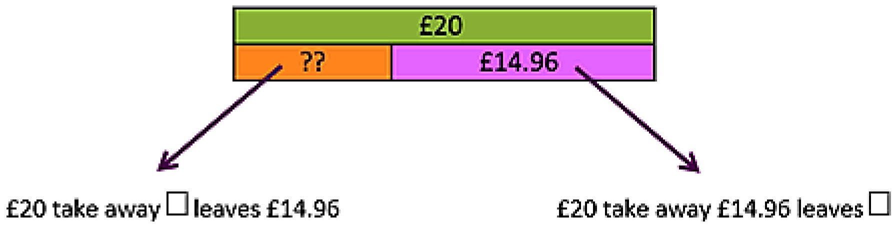




The pictures are different because the models are different: the first one is "subtraction representing reduction" and the second is "subtraction representing comparison". The different models always give numerically the same answer, so long as we are precise about interpreting "difference between" as "*from* the subtrahend (the second number in the subtraction) to the minuend (the first)". *(footnote explains the need for this precision)

When answering the SATS question, the first procedure is interpreting the subtraction as 'taking away', and the second is interpreting the subtraction as the 'difference between'. This is deeper and more powerful than might at first appear: the pupils are reasoning, probably without realising, that

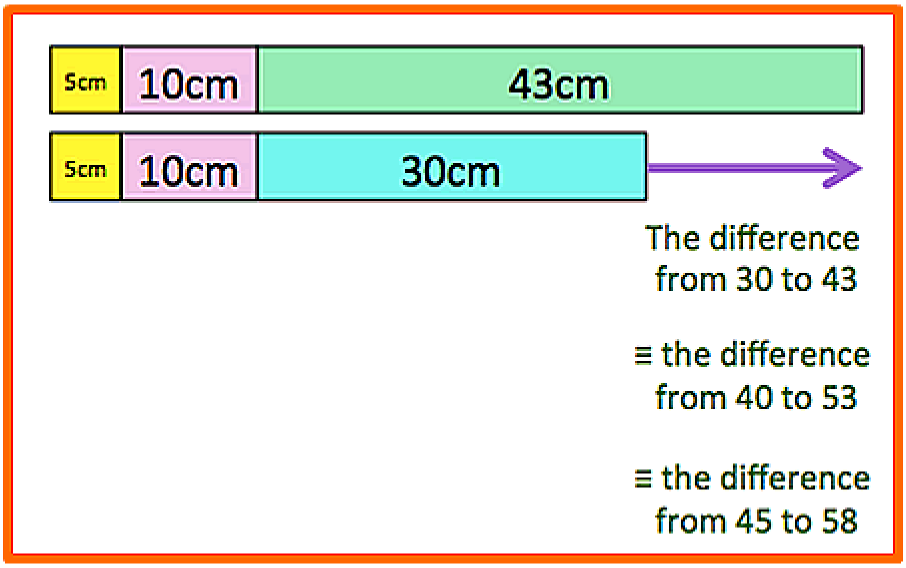
- this is a 'taking away' problem, because Amina had £20 and the shopkeeper took away some of it in exchange for stamps, and Amina was left with £14.96,
- so the problem we need to solve is "£20 take away ☐ leaves £14.96, how much does ☐ represent?"
- but this is conceptually the same as '£20 take away £14.96 leaves ☐, how much does ☐ represent?"



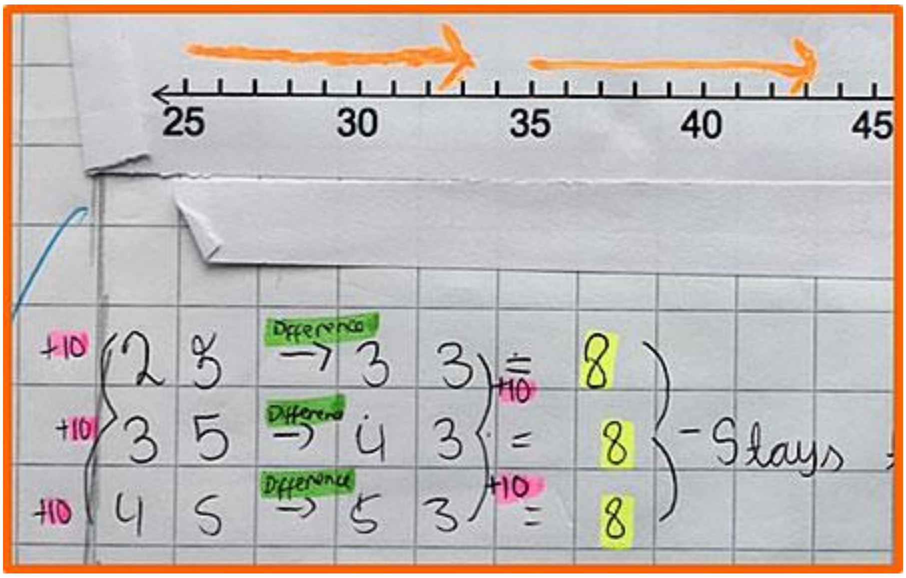
- and that is numerically but not conceptually the same as "the difference between £14.96 and £20"
- which I can work out by starting with £14.96 and augmenting that number by 4p and then £5 until I reach £20
- so the difference between £14.96 and £20, from £14.96 to £20, is £5.04
- so £20 take away £14.96 is also £5.04.

Notice that we are switching between models: we are working out a 'take away' by thinking of it as a 'difference between, from subtrahend to minuend'. Similarly, we can reason that:

58 'take away' 45 ≡
53 'take away' 40 ≡
43 'take away' 30
without working out that each subtraction equals 13:



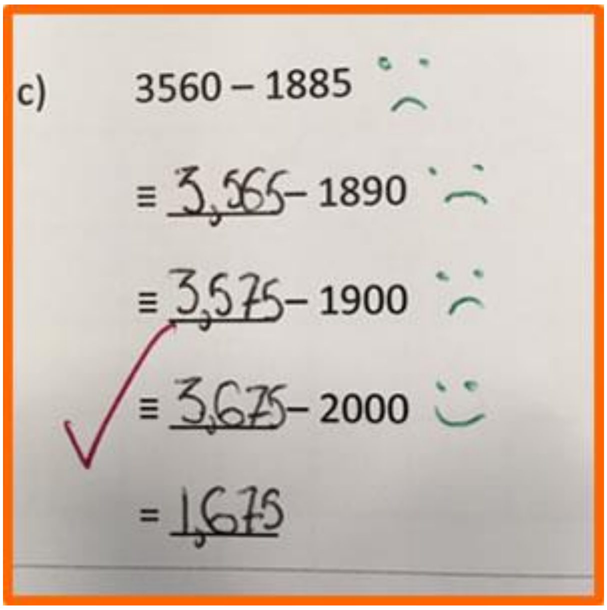
This is **The Principle of Constant Difference**: in a subtraction, if the minuend and the subtrahend both increase or decrease by the same amount, the difference between them (from the subtrahend to the minuend) stays the same.

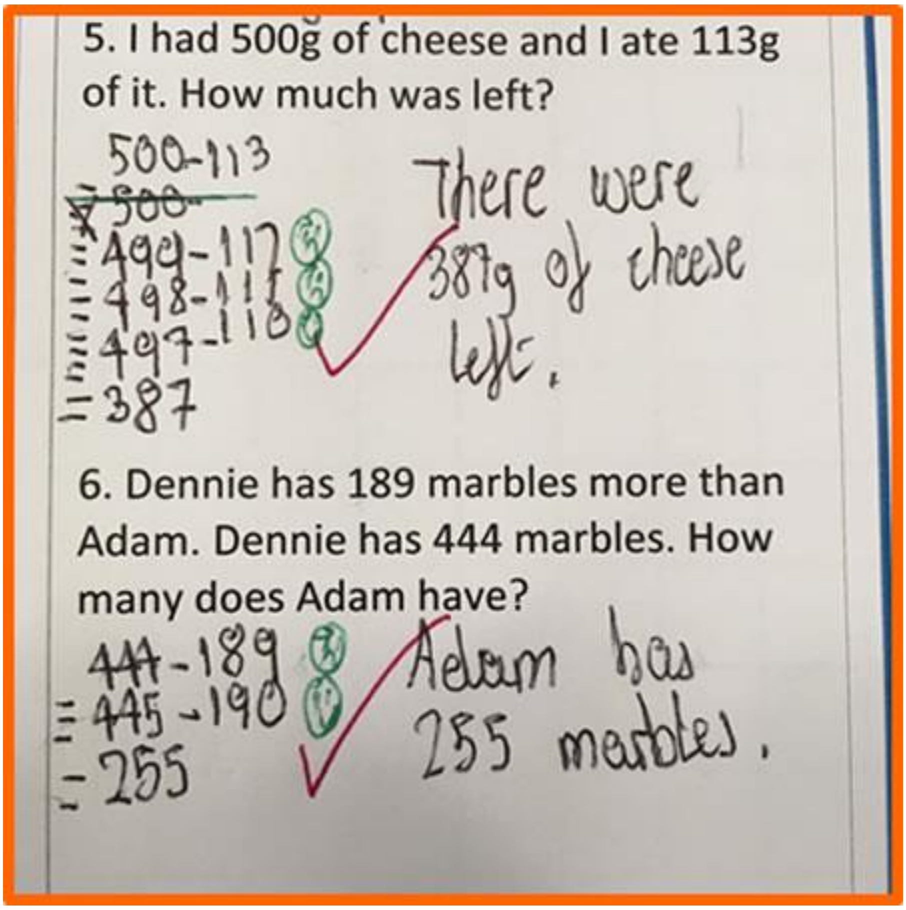


To return to Amina's stamps: the Principle of Constant Difference is an efficient way to calculate

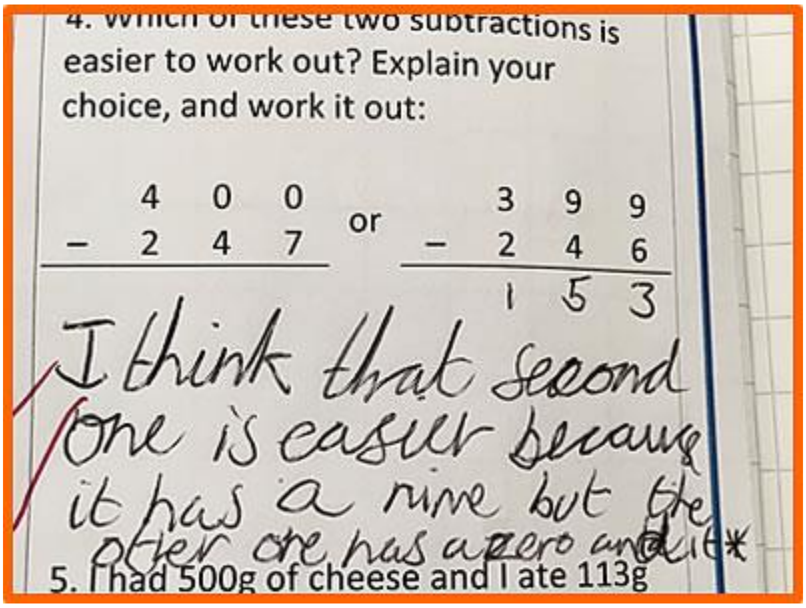
- £20.00 – £14.96 'take away' has the same numerical answer as 'difference between'
- ≡ £20.01 – £14.97 because the minuend and the subtrahend both increase by 0.01
- ≡ £20.02 – £14.98 because the minuend and the subtrahend both increase by 0.01
- ≡ £20.03 – £14.99 because the minuend and the subtrahend both increase by 0.01
- ≡ £20.04 – £15.00 which is easy to work out: a 'nasty' subtraction has become 'nice'
- = £5.04.

These Y5 pupils are reasoning in the same way:

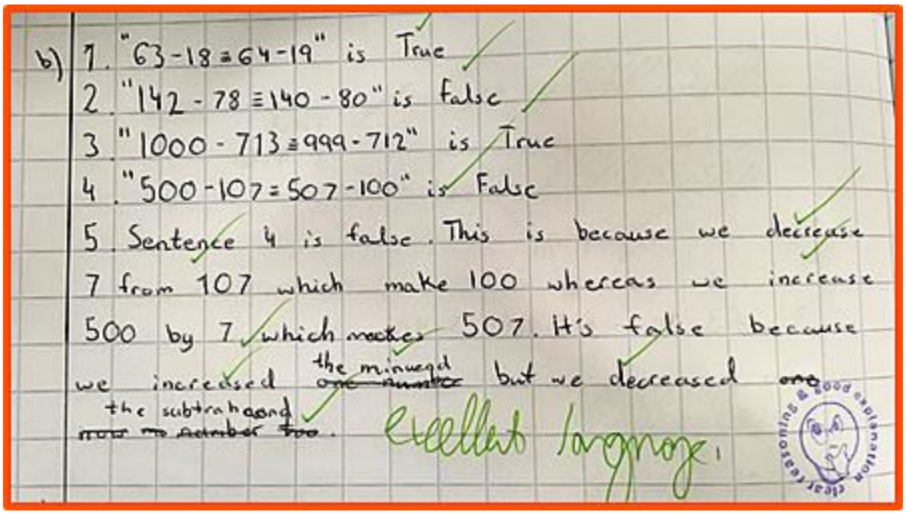




This pupil has justified his choice between calculating 400 – 247 and 399 – 246:



This Y7 student is reasoning about four subtractions, rather than working them out explicitly: the activity is developing her conceptual understanding rather than her calculation fluency.



The power of the Principle is that it extends naturally to subtractions that are procedurally more demanding and / or conceptually more challenging: in particular, subtractions with non-integer terms in KS2, and then subtractions in KS3 with negative subtrahends, and those with algebraic terms in the minuend, the subtrahend, or both:

- 5.3 – 2.7 ‘take away’ has the same numerical answer as ‘difference from … to …’
- ≡ 5.6 – 3 because the minuend and the subtrahend both increase by 0.3
- = 2.6 which is easy to work out: a ‘nasty’ subtraction has become ‘nice’

and then

- 8 – -2 ‘take away’ has the same numerical answer as ‘difference between’
- ≡ 9 – -1 because the minuend and the subtrahend both increase by 1
- ≡ 10 – 0 because the minuend and the subtrahend both increase by 1
- = 10 which is easy to work out: a ‘nasty’ subtraction has become ‘nice’

or

- -5 – -13 ‘take away’ has the same numerical answer as ‘difference between’
- ≡ -2 – -10 because the minuend and the subtrahend both increase by 3
- ≡ 0 – -8 because the minuend and the subtrahend both increase by 2
- 8 – 0 because the minuend and the subtrahend both increase by 8
- = 8 which is easy to work out: a ‘nasty’ subtraction has become ‘nice’

and then, later still

- -9x – -3x
- ≡ -8x – -2x because the minuend and the subtrahend both increase by x
- ≡ -7x – -x because the minuend and the subtrahend both increase by x
- ≡ -6x – 0 because the minuend and the subtrahend both increase by x
- = -6x 'nasty' has become 'nice'

The operation of subtraction occurs in every key stage: the youngest learners subtract small positive integers from slightly larger ones, and A level Further Maths pupils subtract the arguments of complex numbers and wonder what the connection is with logarithms. If learners are to develop confident, flexible and secure understanding of subtraction, they need to encounter it in a conceptually and procedurally continuous way, in every key stage. That can only happen if teachers in each key stage communicate with each about how they are teaching subtraction: the procedures their pupils are using, and the language they are using to describe what those procedures are representing.

The Principle of Constant Difference is procedurally powerful and conceptually rich, and is simple enough that it can be – I would say it should be – explored and grappled with and used throughout the 'middle years', i.e. from Y5 to Y8. This would ensure that pupils do indeed encounter subtraction in a conceptually and procedurally continuous way from primary to secondary. For this to happen, though, primary and secondary teachers need to be

- confident themselves with justifying and using the Principle;
- confident that their 'feeder primary' or 'destination secondary' colleagues are justifying and using it too;

and for *this* to happen, there needs to be regular cross-phase communication between subject leaders – or, even better, cross-phase professional development, so that primary and secondary teachers come together and learn together, as recommended by the Education Endowment Foundation:

> **Are primary and secondary schools developing a shared understandings of curriculum, teaching, and learning?** Both primary and secondary teachers are likely to be more effective if they are familiar with the mathematics curriculum and teaching methods outside of their age-phase.

Creating the opportunity for, and the culture of, dialogue and development between primary and secondary teachers is one of the key aims of the Maths Hubs project 'Improving Continuity Between Primary and Secondary School'. To find out more and get involved, contact your local Maths Hub directly, which you can do via www.mathshubs.org.uk.

*Footnote: The precision, that **difference between**, means **difference from** the subtrahend to the minuend, becomes necessary when the abstract subtractions become less easy, or less natural, to interpret as 'take-away' in the concrete, particularly when the subtrahend is negative (see 3rd and 4th calculations below):

- Why does 9 subtract 3 = 6? Because the difference between 3 and 9, from 3 to 9, is 6.
- Why does -9 subtract 3 = -12? Because the difference between 3 and -9, from 3 to -9, is -12.
- Why does 9 subtract -3 = 12? Because the difference between -3 and 9, from -3 to 9, is 12.
- Why does -9 subtract -3 = -6? Because the difference between -3 and -9, from -3 to -9, is -6.

A model of temperature change leads to the same numerical answers:

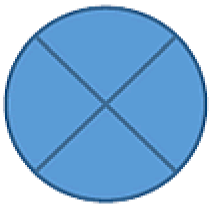
- if the thermometer yesterday recorded 3°C and today records -9°C then the temperature change from 3°C to -9°C is a fall of 12°C, which we can write as -12°C;
- if yesterday it recorded -3°C and today it records -9°C, then the temperature change from -3°C to -9°C is a fall of 6°C, which we can write as -6°C.

Continuity and development in the teaching of fractions across Key Stages 2 and 3

This article looks at the fundamental understanding of what a fraction is and how it works. These are the concepts that children should be engaging with in Primary School and into KS3, as building blocks for sound reasoning throughout their mathematical lives. For each strand of understanding fractions, described below, there is an example question intended to inspect the strength of understanding, by requiring that children use their knowledge in a different way to 'standard' sorts of questions.

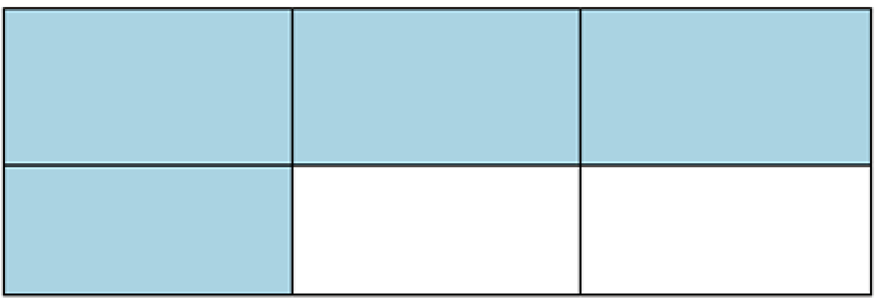
Understanding what fractions are, is crucial for reasoning about fractions as *numbers*, and *within calculations*.

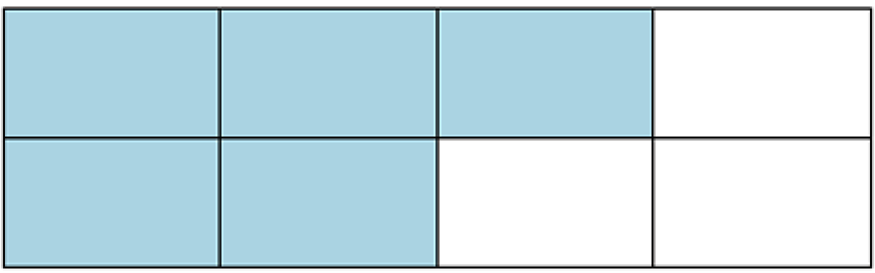
In Upper Key Stage 2, pupils should have a deep understanding of what a fraction is and what the numbers in a fraction represent. The teaching should expose the fact that *the denominator is the number of equal parts in the whole.*



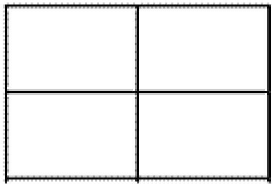
$\frac{\square}{4}$

and that *when comparing and combining fractions, the whole has to be the same* for all the fractions being considered:

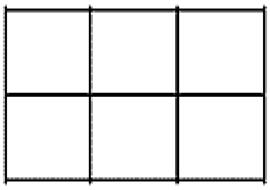




The pupils should be able to make and explain the generalisation that, *for the same whole, the bigger the denominator, the smaller the size of the pieces…*



$\frac{\square}{4}$



$\frac{\square}{6}$

They will also understand that *the numerator tells us how many of those equal parts we have*.

For the same whole, larger denominator = smaller pieces

The following question is one which would be classified as a dòng nǎo jīn question* as it is unfamiliar to the pupils and, unlike most of the other examples they will have encountered, does not ask for an exact answer but is interested in the reasoning that the pupils would have to go through to ensure that the answer is as small as possible.

Using the numbers 5 and 6 only once, make this sum have the smallest possible answer:

$$\frac{\square}{15} + \frac{\square}{10} =$$

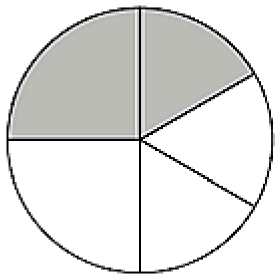
from the [NCETM Year 5 Assessment Materials (Demonstrating Mastery)]

The pupils would have to draw on their understanding that, for the same whole, fifteenths are smaller than tenths, so that the smaller number (5) is put with the tenths.

An understanding of fractions that allows adding with different denominators

As pupils end their time at primary school, their knowledge and understanding of fractions will include basic calculation with fractions. They are expected to be able to use their knowledge of equal parts, and the relation to the whole, to handle fractions with different denominators, when adding and subtracting. This example from the 2017 KS2 SATs shows how representation is used to reveal the structure of calculation and to expose the need for common denominators to calculate with fractions of different denominations.

23 In this circle, $\frac{1}{4}$ and $\frac{1}{6}$ are shaded.



What fraction of the whole circle is not shaded?

2 marks

from the 2017 KS2 SATs Reasoning Papers

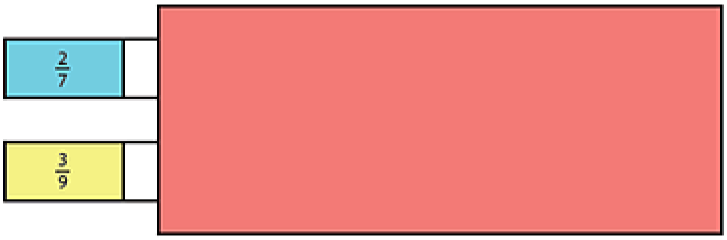
Pupils might be able to reason that the unshaded part of the circle is a half and another half of a sixth, which they should recognise as a twelfth. It is therefore possible that pupils would be able to reason their way to an answer of seven twelfths without having converted the quarter and sixth in to twelfths themselves. This flexibility of thinking and ability to make connections will enable subsequent teachers to build on this reasoning with the conventions of converting fractions into common denominators and record the problem entirely symbolically. This representation also demonstrates that it is not necessary to use 24ths as a common denominator but that the lowest common multiple of 4 and 6 is 12.

The use of images and representations such as this is now very common in Primary Schools across the country and provides a good starting point for Secondary Schools who quickly need to find out where their new intake are at with their maths. If the procedures are not secure or remembered then the representations will allow the pupils to demonstrate their understanding of fractions, onto which more formal methods can be built.

Understanding of the whole in relation to each fraction

Understanding of the whole in relation to each fraction is vital. In this case, pupils need to understand what the fraction represents in each case.

Only a fraction of each whole rod is shown. Using the given information, identify which whole rod is longer



Explain your reasoning.

from the *Year 6 NCETM Assessment Materials – Greater Depth*

This time the shaded portions are the same size and they are asked to consider the whole in relation to the parts. If understanding of parts and wholes and the structure of a fraction is secure, then the pupils should be able to reason that $\frac{3}{9}$ is the same as $\frac{1}{3}$ and therefore there will be 3 of the yellow sections in the whole rod. $\frac{2}{7}$ is less than $\frac{1}{3}$ (or $\frac{2}{6}$) therefore there will be '3 and a bit' blue sections in the whole so the blue rod will be longer. Reasoning their way to an answer in this way would perhaps demonstrate a greater depth of understanding as it is more efficient than using common a denominator of 63.

Fractions as values in their own right

By the time pupils move into Key Stage 3, 'Fractions' no longer has a heading of its own in the curriculum. 'Fractions' is part of 'Number' so understanding their behaviour as values in their own right is key.

This activity develops pupils' ability to reason about fractions as dividends and divisors within calculations. Dividing one fraction by another is the classic algorithm which many adults will quote to you as: 'turn the second one upside down and multiply' as Alice says below:

Alice, Bekah and Clare are explaining why $\frac{2}{3} \div \frac{1}{3} = 2$

- Alice says "Because you turn the second number upside down and multiply, so $\frac{2}{3} \div \frac{1}{3} = \frac{2}{3} \times \frac{3}{1} = \frac{6}{3} = 2$
- Bekah says "Because if I share two thirds of a cake between one third of a person then to get a whole person I need to multiply by three, so that means that the person gets six thirds of the cake and six thirds is the same as two."
- Clare says "$\frac{2}{3} \div \frac{1}{3}$ means 'how many one thirds are there in two thirds?' Because two thirds is the same as $2 \times \frac{1}{3}$, the answer must be 2"

Which explanation do you find most convincing? Why?

from the *NCETM KS3 Assessment Materials*

If pupils understand the structure of division, they can then apply it to the case when a fraction is divided by another. This understanding means that they do not need to remember the rule, they stand a better chance of ensuring that their answer is correct, and also being able to reason their way through a problem where the ability to calculate may not be the focus.

Pupils being able to apply their deep understanding of a subject, to reason and solve unfamiliar problems is what we are looking for when we say that a child has achieved 'Mastery'.

* In teaching for mastery in Shanghai, the dòng nǎo jīn question is a regular part of the lesson that can be something tricky or puzzling about the concept, particularly that encourages pupils to use or develop their understanding in a different way.