Core concept 1.4: Simplifying and manipulating expressions, equations and formulae

This document is part of a set that forms the subject knowledge content audit for Key Stage 3 maths. The audit is based on the NCETM Secondary Professional Development materials and there is one document for each of the 17 core concepts. Each document contains audit questions with check boxes you can select to show how confident you are (1 = not at all confident, 2 = not very confident, 3 = fairly confident, 4 = very confident), exemplifications and explanations, and further support links. At the end of each document there is space to type reflections, targets and notes. The document can then be saved for your records.

1.4.1 Understand and use the conventions and vocabulary of algebra, including forming and interpreting algebraic expressions and equations

How confident are you that you understand and can follow algebraic notation conventions to communicate mathematically including being able to generalise?

1 ☐ 2 ☐ 3 ☐ 4 ☐

How confident are you that you know and can explain the meaning of and identify: term, coefficient, factor, product, expression, formula and equation?

1 ☐ 2 ☐ 3 ☐ 4 ☐

How confident are you that you know and can explain that a letter can be used to represent a value and how substituting a particular value into a generalised algebraic expression changes the value of the expression?

1 ☐ 2 ☐ 3 ☐ 4 ☐

One of the ways in which students interpret algebraic expressions and equations is to work from the general to the particular. For example, to interpret the meaning of an algebraic statement, such as $3x + 5$ or $x^2 - 2$, it is important that students consider these questions:

- ‘How does the value of the expression change as the value of x changes?’
- ‘When does the expression take a particular value?’

Students should realise that there is a difference between situations where a letter represents a variable that can take any value across a certain domain and where, because of some restriction being imposed (e.g. $3x + 5 = 7$, $x^2 - 2 = 9$ or $3x + 5 = x^2 - 2$), it has a particular value (which may be as yet unknown).

An **algebraic expression** is formed from letter symbols and numbers, combined with operation signs and brackets. Each part of an expression is called a term. In the expression $3n + 5$ the terms are $3n$ and $+ 5$.
A formula is an equation linking sets of physical variables. For example, the formula $v = u + at$ has four variables v, u, a and t related by the formula. If the values of three variables are known, the fourth value can be calculated.

An **equation** is a mathematical statement showing that two expressions have equal value. The expressions are linked with the symbol **=**. For example, in the equation $5x + 4 = 2x + 31$, x is a particular unknown number.

A **coefficien**t is a factor of an algebraic term. For example:

- in the term $4xy$, 4 is the numerical coefficient of xy but x is also the coefficient of $4y$ and y is the coefficient of $4x$
- in the quadratic equation $3x^2 + 4x - 2$ the coefficients of x^2 and x are 3 and 4, respectively.

Factor: when a number, or polynomial in algebra, can be expressed as the product of two numbers or polynomials, these are **factors** of the first. For example:

- 1, 2, 3, 4, 6 and 12 are all factors of 12 because $12 = 1 \times 12 = 2 \times 6 = 3 \times 4$
- $(x - 1)$ and $(x + 4)$ are factors of $(x^2 + 3x - 4)$ because $(x - 1)(x + 4) = (x^2 + 3x - 4)$.

A **product** is the result of multiplying one number by another. For example, the product of 2 and 3 is 6 because $2 \times 3 = 6$.

Further support links

- NCETM Secondary Professional Development materials: 1.4 Simplifying and manipulating expressions, equations and formulae, pages 14–17
- More definitions are available in the NCETM Mathematics glossary for teachers in Key Stages 1 to 3

1.4.2 Simplify algebraic expressions by collecting like terms to maintain equivalence

How confident are you that you know and can explain how to simplify an expression by identifying and collecting like terms?

1 ☐ 2 ☐ 3 ☐ 4 ☐

Students should see the process of 'collecting like terms' as essentially about adding things of the same unit. Younger students are often excited by the fact that calculations such as 3 000 000 + 2 000 000 are as easy as 3 + 2. Later, they realise that the same process is at work with equivalent fractions. For example:

$$\frac{3}{7}+\frac{2}{7}=\frac{5}{7}$$

Students begin to generalise this to 3 (of any number) + 2 (of the same number), and finally to symbolise this as $3a + 2a$.

Teaching approaches that are solely procedural and do not help students understand the idea of unitising and the important principle that letters stand for numbers and not objects, should be avoided. For example, it is incorrect to teach that $3a + 2a = 5a$ because 'three apples plus two apples equals five apples' and this approach (often termed 'fruit salad algebra') should be avoided.

Students should fully appreciate that 'collecting like terms' is not a new idea but a generalisation of something they have previously experienced when unitising in number. They should understand what like terms ***are*** and ***are not*** and should experience a wide range of standard and non-standard examples (for example, constant terms, terms containing products, indices, fractional terms). Students should come to realise that when they are simplifying algebraic expressions such as $2xy + 5xy$ as $7xy$, they have obtained an equivalent expression (i.e., one with exactly the same value even though it has a different appearance).

1.4.3 Manipulate algebraic expressions using the distributive law to maintain equivalence

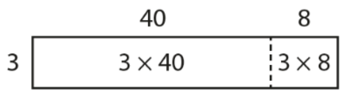
How confident are you that you know and can explain how to apply the distributive law to expand expressions including more complex situations?

1 ☐ 2 ☐ 3 ☐ 4 ☐

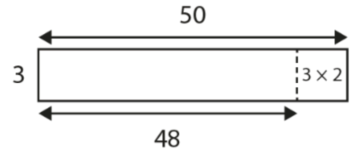
How confident are you that you know and can explain how to apply the distributive law to factorise expressions including more complex situations?

1 ☐ 2 ☐ 3 ☐ 4 ☐

Students will have learnt at Key Stage 2 that to calculate an expression such as 3 × 48 they can think of it as 3 × (40 + 8), which equals 3 × 40 + 3 × 8. Students may know this as the distributive law, although this should not be assumed. It is important at Key Stage 3 that students come to see this as a general structure that will hold true for all numbers. They should be able to express this general structure symbolically (i.e., $3(a + b) = 3a + 3b$), and pictorially, for example by using an area model:



Students should also be able to generalise this further to subtraction (i.e., $3(a - b) = 3a - 3b$) by considering a calculation, such as $3 \times 48 = 3(50 - 2) = 3 \times 50 - 3 \times 2$, and an area model, such as this:



It is useful at this stage to draw attention to the 'factor × factor = product' structure of the equivalence $3(a + b) = 3a + 3b$: two factors, 3 and $(a + b)$, have been multiplied together to give a product equivalent to $3a + 3b$. This will support students' understanding of the inverse process of factorising; for example, 'If the product is $3a + 3b$, what might the two factors be?'.

To gain a deep and secure understanding, students will benefit from experiencing a wide range of standard and non-standard examples (such as negative, decimal and fractional factors, including variables). Careful attention to the use of variation when designing examples will support students to generalise.

Factorising is breaking up the expression into factors, so that when you multiply the factors together you obtain the original expression.

The first stage of factorising in algebra is to understand the relationship between factorising and expanding brackets. Factorising is the reverse process to expanding brackets.

Example 1:
Expanding $3(2a + 5)$ gives $6a + 15$.
So factorising $6a + 15$ gives $3(2a + 5)$.
3 is the highest common factor of 6 and 15.
The two factors of $6a + 15$ are 3 and $2a + 5$.

Example 2:
Expanding $4ab(2a + 3b)$ gives $8a^2b + 12ab^2$.
So factorising $8a^2b + 12ab^2$ gives $4ab(2a + 3b)$.
$4ab$ is the highest common factor of $8a^2b + 12ab^2$.
The two factors of $8a^2b + 12ab^2$ are $4ab$ and $2a + 3b$.

Factorising by grouping:

Expanding $(a + b)(c + d)$ gives $ac + ad + bc + bd$.
So factorising $ac + ad + bc + bd$ gives $(a + b)(c + d)$.
In the expression $ac + ad + bc + bd$, pairs of terms with a common factor are grouped together and factorised to give $a(c + d) + b(c + d)$
These two terms have a common factor $(c + d)$.
So can be written in the equivalent form $(c + d)(a + b)$, which can also be written as $(a + b)(c + d)$.

Factorising the difference of two squares:

Expanding $(a + b)(a - b)$ gives $a^2 - ab + ab - b^2 = a^2 - b^2$.
So factorising $a^2 - b^2$ gives $(a + b)(a - b)$.

Factorising expressions of the form $x^2 + bx + c$:

Since $(x + a)(x + b) = x^2 + (a + b)x + ab$, to factorise $x^2 + 6x + 8$, we need to find two numbers whose product is +8 and whose sum is +6. The two numbers are +2 and +4.
So $x^2 + 6x + 8$
$= x^2 + 2x + 4x + 8$
$= x(x + 2) + 4(x + 2)$
$= (x + 2)(x + 4)$

Further support links

- NCETM Secondary Professional Development materials: 1.4 Simplifying and manipulating expressions, equations and formulae, page 6
- NCETM: Using Mathematical Representations at Key Stage 3: Algebra tiles

1.4.4 Find products of binomials

How confident are you that you know, and can explain using representations where necessary, how to find the product of two binomials including the special case where the product is the difference of two squares?

1 ☐ **2** ☐ **3** ☐ **4** ☐

A **binomial** is an algebraic expression of the sum or difference of two terms, for example $3x^2+2x$.

In *1.4.3* (above), students used the distributive law to expand a single term over a binomial. Here they use the same law to work with pairs of binomials. Students should understand that this expansion is a generalisation of the familiar 'grid method' for multiplication. For example, the layout below (left) representing $(2x + 4)(3x + 6)$ can be seen as a generalisation of the familiar grid layout (below, right) for 24×36 or $(20 + 4)(30 + 6)$.

	$2x$	4
$3x$	$6x^2$	$12x$
6	$12x$	24

	20	4
30	600	120
6	120	24

The use of algebra tiles to represent this may make more explicit the connection with the area model of multiplication.

The area model will also support students to understand and justify that the product of an expression with, for example, two terms in the first expression and three terms in the second expression, will have six (i.e. 2×3) terms before simplifying. For example, $(2a + 3)(5a + 6y + 4)$ can be represented as:

	$2a$	3
$5a$		
$6y$		
4		

Students need to generalise further to situations where there are more than two binomials and realise that the product of more than two binomials can be reduced to two polynomials by successive multiplication of pairs. For example, the product $(a + b)(a + 3b)(a - b)$ can be reduced to the product of two polynomials by combining any two binomials. It will be important to introduce examples where alternative approaches might be more efficient and/or elegant, and to give students the opportunity to discuss these. For example, $(a + b)(a + 3b)(a - b)$ can be transformed into $(a^2 + 4ab + 3b^2)(a - b)$ and then multiplied out further. Alternatively, it could be transformed into $(a^2 - b^2)(a + 3b)$ by noticing that the first and last factors produce the difference of two squares.

Further support links

- NCETM Secondary Professional Development materials: 1.4 Simplifying and manipulating expressions, equations and formulae, pages 24–27

1.4.5 Rearrange formulae to change the subject

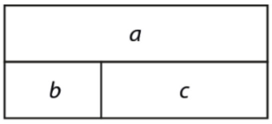
How confident are you that you know, and can explain how to rearrange a formula to change the subject through deep understanding of additive and multiplicative operations and inverses?

1 ☐ 2 ☐ 3 ☐ 4 ☐

At Key Stages 1 and 2, students had experience of expressing number relationships in different ways. So, for example, if students know $3 + 4 = 7$, they should also know the 'three facts for free': $4 + 3 = 7$, $7 - 4 = 3$ and $7 - 3 = 4$. Similarly, students should be aware that $3 \times 4 = 12$ gives rise to $4 \times 3 = 12$, $12 \div 3 = 4$ and $12 \div 4 = 3$. At Key Stage 3, students extend this knowledge to equations, understanding that the same relationship can be expressed in different ways.

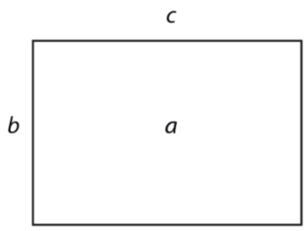
Students should distinguish between additive and multiplicative structures. Additive structures can be shown clearly by a bar model. For example, $a = b + c$ can be represented as:

This gives rise to the following equivalent expressions: $a = b + c$; $a = c + b$; $a - b = c$; $a - c = b$.



Students need to be aware that this additive structure can also be applied to more complex equations. For example, $(x^2 + a) + (x^3 - px + m) = (4 - p)$ can be rewritten as $(x^2 + a) = (4 - p) - (x^3 - px + m)$, which – because the left-hand side is also an additive expression – can be written as $a = (4 - p) - (x^3 - px + m) - x^2$ to make ***a*** the subject.

When considering multiplicative structures, an area model is helpful to reveal the relationships. For example, $b \times c = a$ can be represented as:



Students can then see the equivalent expressions: $b \times c = a$; $c \times b = a$; $a \div c = b$; $a \div b = c$.

When working with formulae, students should appreciate that, when expressing the relationship between one variable (the subject of the formula) and the rest of the expression, it is possible to evaluate any of the variables, given values for all the others. For example, $F = \frac{9}{5}C + 32$ and $C = \frac{5}{9}(F - 32)$ allow for different values to be calculated and offer different perspectives of the relationship between degrees Fahrenheit (F) and degrees Celsius (C). Students should appreciate that the process of changing the subject of a formula is essentially the same process as solving an equation in one unknown.

Further support links

- NCETM Secondary Professional Development materials: 1.4 Simplifying and manipulating expressions, equations and formulae, pages 28–29

Notes

