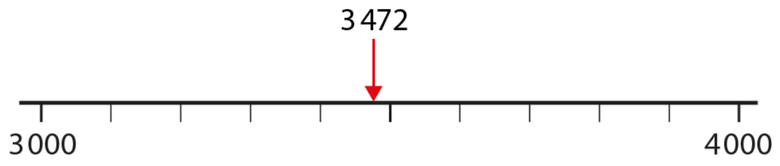
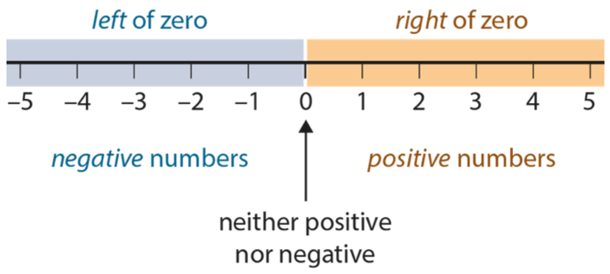
Mastery Professional Development

Mathematical representations

Single number lines

Guidance document | Key Stage 3

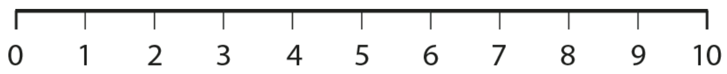




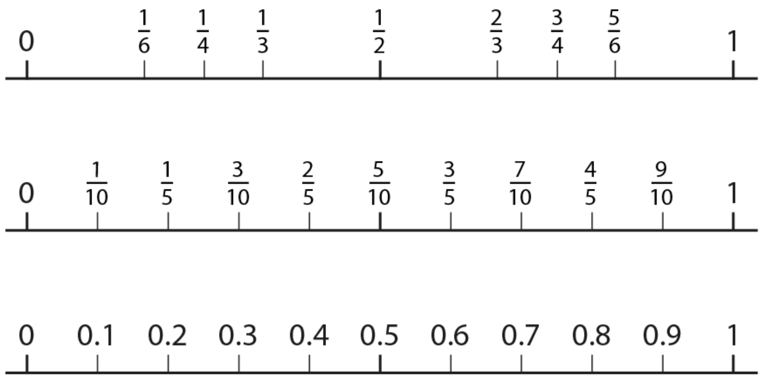
Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

What they are

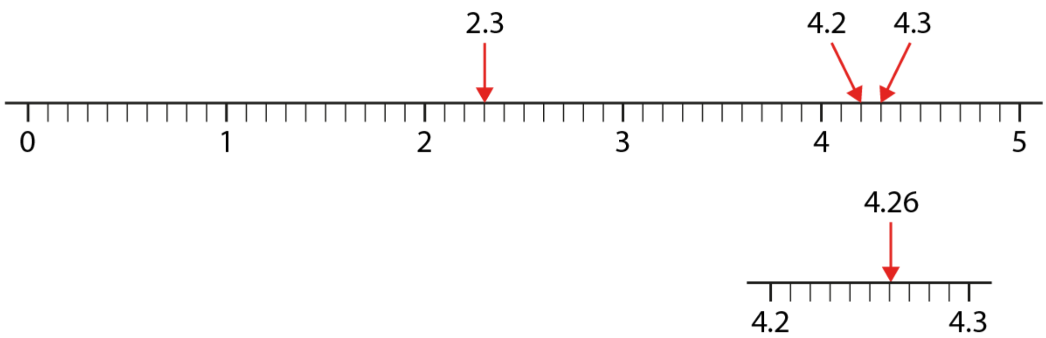
Single number lines are straight lines with numbers evenly spaced, in order, along their length. While vertical representations of the number line support the link to reading scales and the *y*-axis, single number lines are usually horizontal, with numbers increasing from left to right and decreasing when moving from right to left. The number line is not a device for measuring but a representation to support students' understanding of the number system.



In contrast to concrete materials, such as cubes and Dienes, which offer a **cardinal image** of number (i.e. a representation of quantity), number lines provide an **ordinal image** (i.e. a representation of position). The divisions on a number line can take any value, and so fractions and decimals can be labelled and the relationships both within and between each set of numbers can be considered.

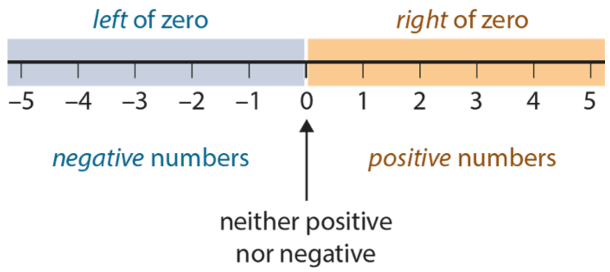


The gap between any two numbers on a number line can be examined, and students' structural understanding of the number system (in terms of tenths, hundredths, etc.) can be explored.



By considering where numbers such as 2.3 and 4.26 might be placed on the number lines above, for example, students can be supported to understand the important idea of 'bounded infinity' (i.e. that between any two numbers there are always other numbers).

The number line can be extended infinitely in both directions (the idea of 'unbounded infinity') and so can be used to represent positive and negative numbers.



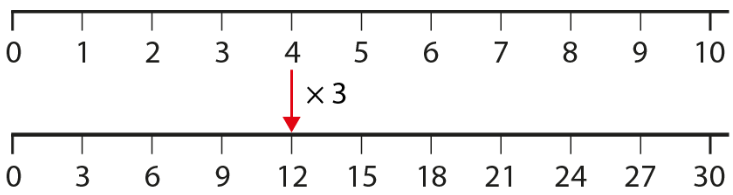
As well as providing a representation of numbers, number lines also provide a representation of number operations. Addition and subtraction can be modelled as counting on (augmentation) and counting back (reduction) from a fixed starting point, and subtraction can also be conceptualised as the difference (or gap) between two numbers. While number lines support the understanding of aggregation (combining two or more parts to make a whole) and partitioning (breaking down a whole into two or more parts) less well than some other representations (bar models, for example), their ability to reveal the structure of a calculation makes them a useful representation of problems involving addition and subtraction.

Why they are important

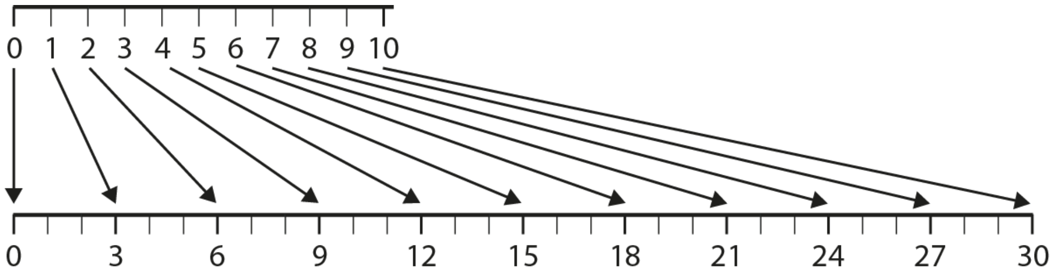
Single number lines provide an important representation of the number system. At both Key Stages 2 and 3 they are particularly useful for thinking about rounding (to the nearest whole number or number of decimal places / significant figures), as they provide a sense of how close other numbers are to the number being rounded. The number line can be used to think about what is happening when we round numbers, in a way that many other representations struggle to capture, and this can also be extended to the concept of bounds, such as those arising from the inaccuracy of measurement, for example.

By zooming in on portions of a number line, to reveal tenths between the integers, hundredths between the tenths and so on, number lines support students in developing a deeper understanding of decimals. One of the major benefits of the number line is that it highlights the connection 'between' numbers and indicates the continuous nature of number.

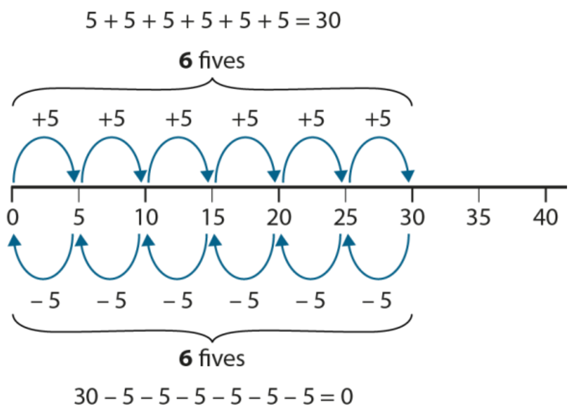
Concrete resources can be very effective in supporting students' understanding of cardinal numbers, in understanding addition as the act of collecting two sets together and subtraction as the removal of one set from another. Number lines provide an important alternative ordinal image of both number and number operations and support students' understanding of addition as counting on and subtraction as counting back and difference. Multiplication can be conceptualised as repeated addition (equal jumps on a number line), or as a change of counting unit:



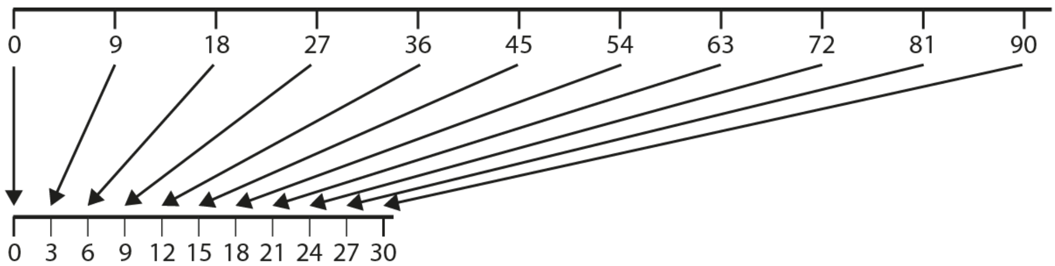
Alternatively, by visualising a number line being stretched, the important sense of multiplication as scaling can be introduced.



Division can be represented as repeated subtraction (the quotitive model of division):



Alternatively, compressing or scaling down a number line by a factor of three, for example, can be used to represent dividing by three or multiplying by $\frac{1}{3}$.

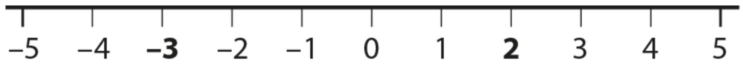


How they might be used

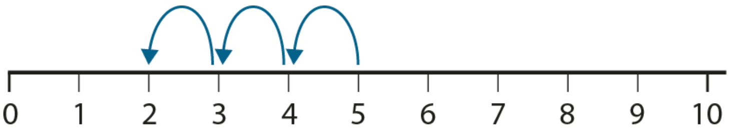
Negative numbers

During Key Stage 2, students encountered negative numbers in simple number problems and real-life contexts.

At Key Stage 3, students should continue to develop their understanding of negative numbers. Number lines can be particularly helpful in supporting students as they re-evaluate their understanding of 'less than' and 'greater than' in contexts involving negative numbers. A common misconception is that negative numbers with a larger magnitude than positive numbers have a greater value; for example, that $-3 > 2$. Positioning the numbers -3 and 2 on a number line can help students to identify that negative numbers are less than natural numbers (as well as recognising that the further away the negative number is from zero, the smaller the value is).

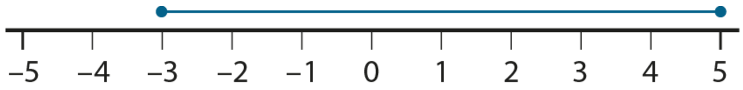


Students often find the idea of adding a negative number difficult, and the number line can help with this. The calculation 5 + (–3), for example, can be thought of as starting at 5 and counting on negative 3.

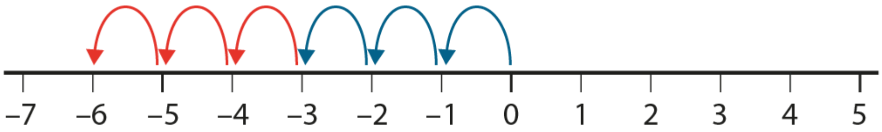


Recognising that '5 count on negative 3' is the same as '5 count back 3' or '5 subtract 3' highlights the equivalence of the addition of negative numbers and subtraction.

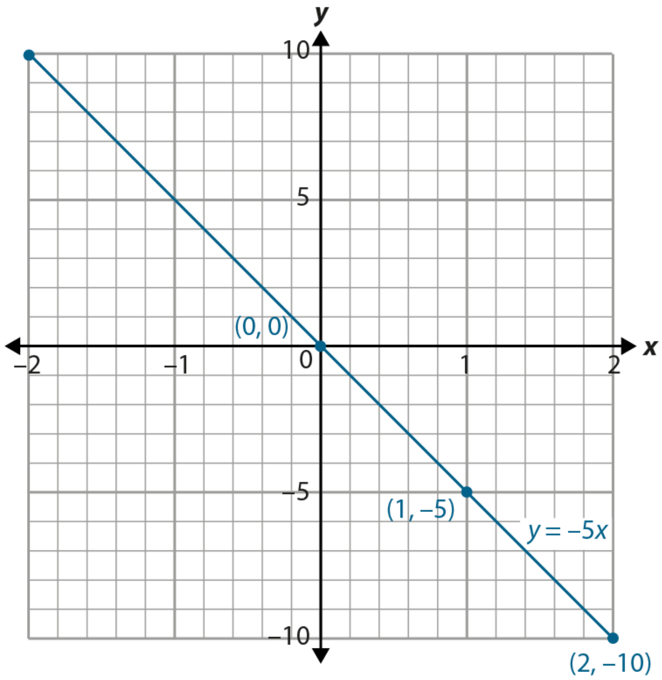
In the same way as for positive numbers, the subtraction of a negative number can be thought of as the difference between the minuend and the subtrahend. For example, when subtracting −3 from 5, the solution is the difference (gap) between 5 and −3, and this can be shown on a number line.



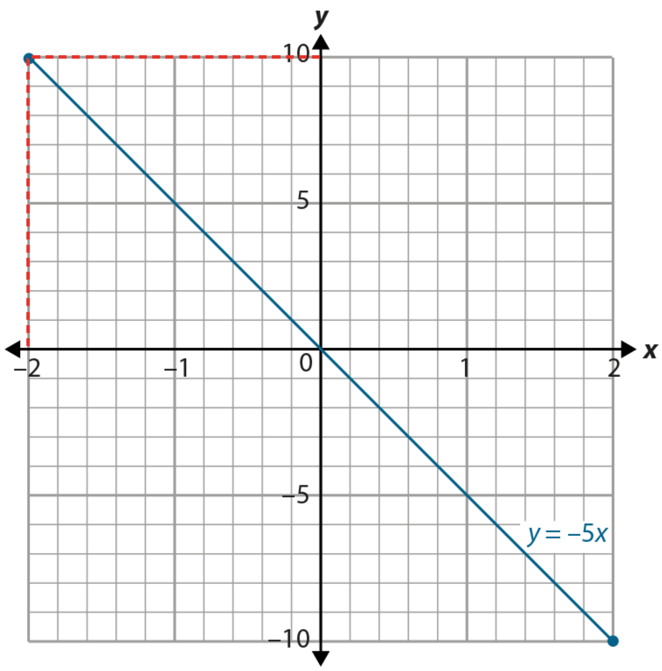
When trying to reason whether an answer is positive or negative when multiplying positive and negative numbers together, confusion can often arise. A number line can be used to demonstrate that, for example, $2 \times (-3) = (-3) + (-3) = -6$, interpreting multiplication as repeated addition.



Extending the use of the number line to multiplying two negative numbers, however, is not necessarily the best use of this representation and an alternative, in the context of graphing equations, might be appropriate. For example, to represent the multiplication $(-5) \times (-2)$ graphically, the number −5 can be represented as the line $y = -5x$. Substituting values for x into '$-5x$' to give the corresponding value for y enables coordinates that lie on the straight line $y = -5x$ to be found. For example, when $x = 1$, $1 \times (-5) = -5$, when $x = 2$, $2 \times (-5) = (-5) + (-5) = -10$ and when $x = 0$, $y = 0 \times (-5) = 0$, giving the coordinates (1, –5), (2, –10) and (0, 0) respectively. These coordinates can be plotted and the line $y = -5x$ drawn.



To multiply by −2, we read off −2 from the horizontal axis, leading to +10 on the vertical axis and showing (−5) × (−2) being equal to +10.



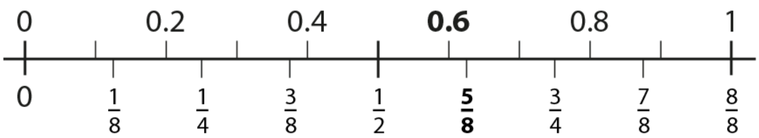
While a number line may provide support for some students as they develop their understanding of negative numbers at Key Stage 3, it is important to recognise when it is no longer the most suitable representation and the use of +1 and −1 algebra tiles, for example, may be more suitable, or indeed the use of any representation may no longer be necessary.

Fractions and decimals

In their work on ordering numbers at Key Stage 2, students will have developed an appreciation of 'unbounded infinity', i.e. that given any number, there will always be a larger or smaller number that can be placed on the number line. An aspect of infinity that students may not have yet met, however, is that of 'bounded infinity'; namely, that given any two numbers, there will always be another number (greater than the smaller and less than the larger) that can be placed between them on a number line.

At Key Stage 3, students develop further their understanding of the different ways that numbers can be expressed, and become more proficient in changing from one form to another. Number lines can be used to develop students' awareness that different representations of a number can reveal something about that number's structure. They also allow students to make comparisons and order numbers.

It is important that students develop methods for ordering fractions and decimals that include converting between fractions and decimals. For example, when determining which is bigger, $\frac{5}{8}$ or 0.6, a number line can be used to support understanding.



It is important that students think carefully about how the number line is labelled and recognise equivalent fractions; for example, that $\frac{2}{8}$ can be labelled as $\frac{1}{4}$ and $\frac{8}{8}$ is equal to one. Being able to convert freely between decimals and fractions (for example, recognising that 0.6 is the same as $\frac{6}{10}$ or $\frac{3}{5}$) and developing their understanding of how numbers can be represented differently, provides students with an appreciation of magnitude and the skills needed to compare and order numbers in a variety of different contexts. Working with number lines in this way supports students in being able to find a number in between any other two given numbers (whether two decimals, two fractions or one fraction and one decimal).

Rounding

It is important, when rounding to an appropriate degree of accuracy, that students recognise that they are trying to find a number, with a specified number of decimal places or significant figures, to which the given number is closer and are not just following an algorithm without understanding.

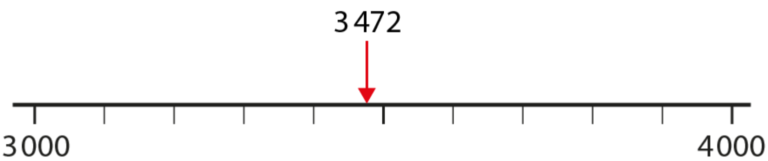
Students may, for example, keep on rounding 3 472 until they achieve a number to one significant figure:

3 472 → 3 470 (because $2 < 5$)

3 470 → 3 500 (because $7 > 5$)

3 500 → 4 000 (as 5 is halfway)

and not realise that 3 472 is closer to 3 000 than 4 000. A number line can be used to support students' understanding.



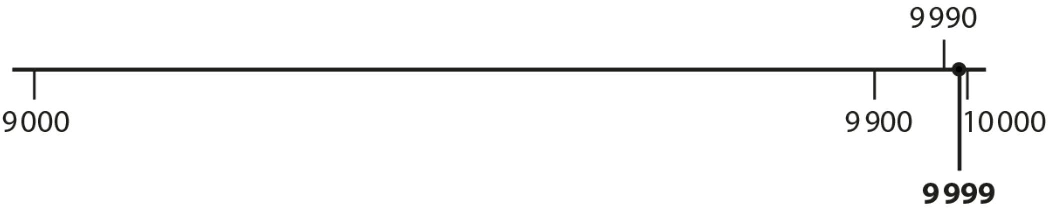
Locating the number to be rounded on the line and identifying the critical values of 3 000 and 4 000 either side, can help students to see that 3 472 is closer to 3 000 and so rounds, to one significant figure, to 3 000 rather than 4 000.

Students can also find it challenging to identify when a zero digit is significant. A number line may help students clearly see which number to round to and avoid confusion. For example, the number 305 contains a zero digit within the number. The results when rounding 305 to one, two and three significant figures are all different.

To round 305 to one significant figure, positioning 305 on a number line between 300 and 400 makes it clear that 305 is much closer to 300 than 400 and so should be rounded down to 300. If students are struggling to recognise the critical values to one significant figure, either side of 305, as 300 and 400, labelling other values on the number line may be helpful.

When rounding to two significant figures, 310 (being the smallest number with two significant figures greater than 305) needs to be added to the line. Students should be able to recognise that 305 rounds to 310 to two significant figures (and that for 301, 302, 303 and 304, rounding to two significant figures would be the same as rounding to one significant figure (300)). Seeing that 305 already has three significant figures and so requires no further rounding relies, to some extent, on recognising that the zero in 305 is significant, as it is between two non-zero digits. However, the recognition that some numbers, like, 301, for example, can only be rounded to one significant figure (and already contain three significant figures) is important, and students should be exposed to a variety of different numbers when rounding, to enable a deep understanding to be developed.

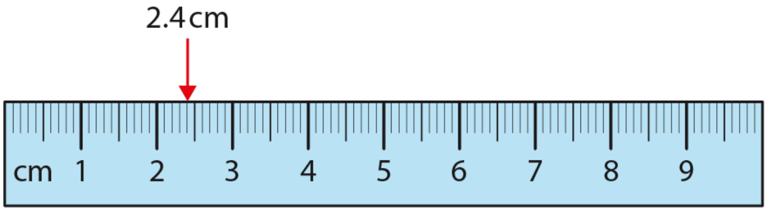
9 999 is an example of a particular idea that students often have difficulties with (i.e. a trail of 9s, which might result in doing lots of sequential rounding up, rather than seeing the size of the number and appreciating that it is 10 000 when rounded to one, two and three significant figures). Positioning 9 999 on a number line will help students see clearly which number to round to and avoid confusion.



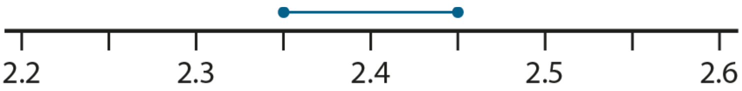
Using number lines in this way can support students in identifying what to do in a variety of different scenarios and how zero digits can be considered. Number lines are an important representation to support the development of a deep understanding of the concept of rounding.

Upper and lower bounds

When measuring quantities, absolute accuracy is never possible, and the level of accuracy can be stated in terms of 'to the nearest…'. For example, a length may be measured as 2.4 cm to the nearest millimetre.



A number line can be used to help students to visualise the possible values that such a measurement might lie between.



The space where values round to 2.4 can be identified, giving 2.35 as the smallest value that would round up to 2.4 and 2.45 as the point at which we would round up to 2.5, to the nearest millimetre. This gives the bounds $2.35 \leq \text{length} < 2.45$, or 2.4 ± 0.05 cm.

While the decision to round the halfway point of 2.35 up to 2.4 might be considered to be an arbitrary one, it is important that students recognise that this is what mathematicians have chosen to do and are consistent in doing it. Using a number line to show the range of values helps to make visible the halfway points between the rounded quantity and the values (to the nearest millimetre) to either side. Once students have represented numerous problems on a number line, they should begin to identify why these halfway points are significant and how they relate to the degree of accuracy that has been used when measuring.

Further resources

One of the features of a single number line, which can aid students' understanding of the continuous nature of number, is the ability to 'zoom in' between any two numbers. While this is possible, to an extent, on a drawn or printed number line, the ability to stretch the number line in order to 'home in' on the gap between two numbers (and explore the concept of bounded infinity) is limited when using a static representation. There are several online resources, however, that make this process more accessible.

See, for example:

Math is Fun
https://www.mathsisfun.com/numbers/number-line-zoom.html
This resource comprises a scrollable number line (by clicking on the left and right arrows), which shows integers by default. Using the 'Zoom In' button and clicking on the number line, stretches the line, increasing the gap between the numbers. As the gap increases, the space between the numbers is divided into tenths, with decimal labels appearing as you zoom further. The ability to zoom continues, with decimal numbers with an increasing number of decimal places becoming visible. The 'Zoom Out' button can be used to zoom out again, and selecting 'Reset' puts the number line back to displaying integers only.

Math Learning Center
https://apps.mathlearningcenter.org/number-line/
This number line contains positive whole numbers by default and while it does not give the option to zoom in between the numbers, it can be customised to display decimals, fractions and negative numbers. The 'Select Numbering' option also allows for the interval between numbers to be selected, for example, displaying even whole numbers only, or thirds, or 0.5, 1.0, 1.5, etc. It is also possible to remove the tickmarks (or change the spacing between the tickmarks via the 'Select Spacing' option) and to have a number line with no numbers. The 'New Custom Tickmark' option allows a moveable tickmark to be added to the top or bottom of the number line for entering custom values. When the 'Hide Line Numbers' option is selected, clicking the spaces below the number line shows (or hides) individual numbers. A scroll bar at the top of the screen allows the number line to be moved back and forth, or the 'Go To' button can be used to jump to locations on the number line beyond the scrolling limits.

When modelling calculations on the number line, there are several options to add arrows to represent jumps along the line. Once an arrow has been placed on the line using the 'New Jump' button, the drag handle can be used to resize the jump or to create a backward jump. There are a number of labelling options for displaying the length of a jump, either automatic labelling, custom labelling, which allows individual values to be entered, or the option to have no labels on the jumps.

Additional features include a keypad for creating expressions and equations, drawing tools and a lasso to select parts of the number line.

CPM Tiles
https://technology.cpm.org/general/tiles/
Selecting 'Number Lines' from the left-hand menu gives the option of dragging a number line into the work area. Arrows at either end of the number line can be used to extend the line in both directions, and there is the option to change the default numbering of ones to any whole number up to and including 100. Double-clicking on the number line swaps the orientation between a horizontal and vertical display. Additional features include a positive and negative tile and a coloured arrow, each with limited functionality.

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