MathsHUBS Resources

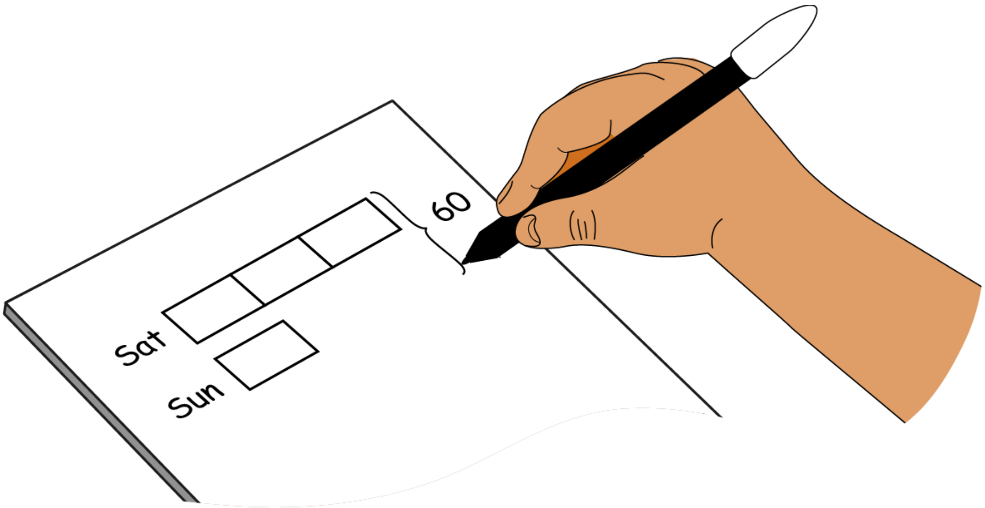
National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

Mathematical representations

Bar models

Guidance document | Key Stage 3

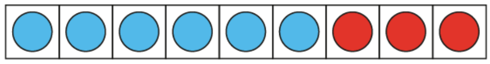


Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

What they are

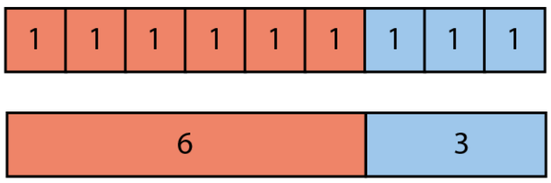
Bar model diagrams use rectangles to represent both known and unknown quantities, and the relationships between them, in mathematical problems. The build-up to using the bar model representation has its beginnings at primary school, with students placing discrete objects in a line to model a problem before progressing to using individual boxes with an object in each box.

$6 + 3 = ?$

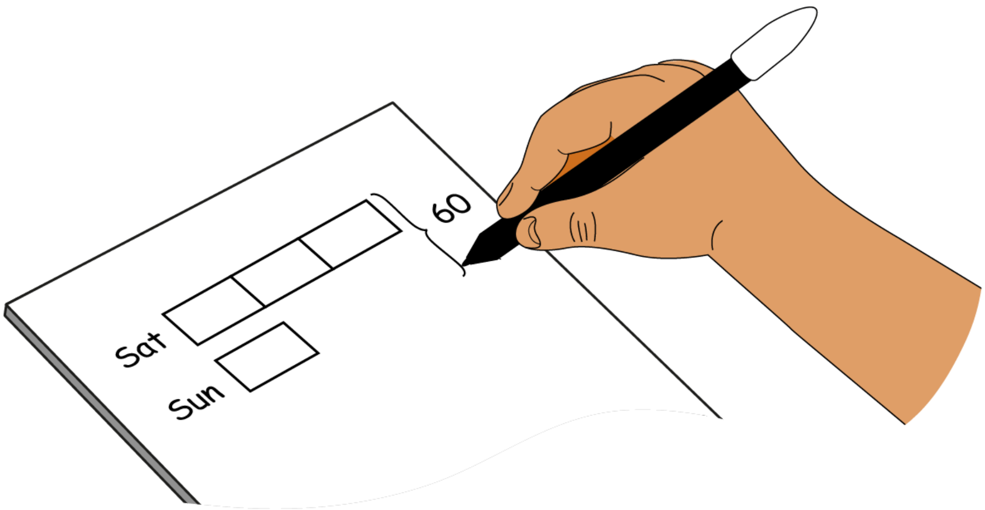


$6 + 3 = ?$

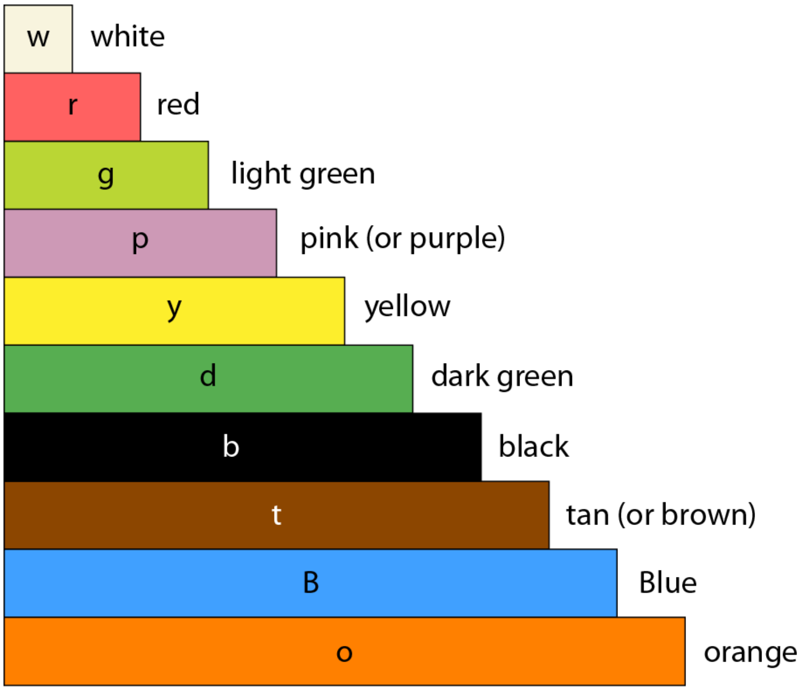
The use of discrete objects can be replaced by the introduction of bars. First, each 'one' can be represented by an individual bar and then the total quantity can be represented (labelled inside the bar).



Strips of paper can be used to represent the rectangles, allowing for some physical manipulation. Alternatively (and more typically at Key Stage 3), bar models are often drawn and used by students as part of their written work when solving problems.



Cuisenaire® rods are one of the most familiar concrete tools for bar modelling. Use of the rods can help transition students from using individual boxes to drawing continuous bar models.

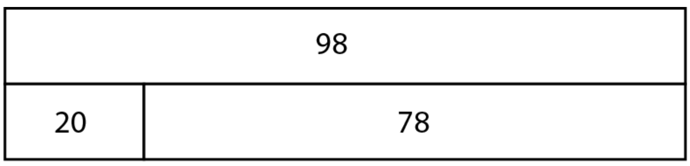


While the rods can take different values, bar models provide the flexibility of being able to draw a 'train' and then amend it, to give a visual picture of what is happening when, say, a calculation is performed. Not needing equipment, other than pencil and paper, also makes bar models versatile and enables them to be used when other equipment, such as Cuisenaire® rods, is unavailable or prohibited.

A bar model diagram can be used to model both additive and multiplicative structures within problems.

1. Part–part–whole

A bar model can be used to represent problems involving the sum of two or more quantities making up a whole.



Whole = sum of parts

98 = 20 + 78

The relationship between the parts and the whole is shown clearly in the bar model diagram above. Not only the sum of 20 + 78, but also the related subtractions, i.e. 98 – 20 = 78 and 98 – 78 = 20. Additionally, the sense of subtraction as 'take away' (i.e. removing the 20 from the whole leaves 78) and as difference (i.e. 78 is the difference between 98 and 20) is also evident. In this way, the part–part–whole model provides a powerful way of viewing addition and subtraction in a relational and connected way.

When drawing a part–part–whole bar model diagram, it is advisable to ensure that the rectangles are proportional to each other. For example, a rectangle representing nine should be roughly one and a half times the length of a rectangle that is representing six.

9

6	?

$$9 = 6 + ?$$

While this is not always possible, for example, when using bars to represent unknowns in equations, the act of redrawing a model, once the value of the variable has been found, can be a useful activity for students. In some representations of the part–part–whole model, the 'whole' bar is replaced with a curly bracket, as in the diagram above.

2. Equal parts

Where problems involve the sum of equal quantities, or quantities being divided into equal parts, a bar model can be used to show these relationships.

Whole		
Part	Part	Part

For example, when thinking about a problem about populations, a bar model diagram can be drawn.

Example problem:

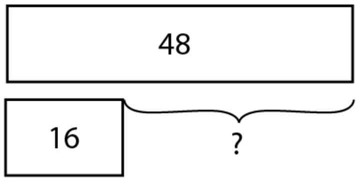
In 2011, about one million people lived in Birmingham, with approximately $\frac{1}{5}$ of the population being over 60 years old. Approximately how many over-60s lived in Birmingham in 2011?

1 000 000				
?	?	?	?	?

The equivalent relationship of multiplying by $\frac{1}{5}$ and dividing by five is a key concept to understand when representing the problem with a bar model. Once the structure of the problem has been identified, a variety of calculation strategies can be used to obtain a solution.

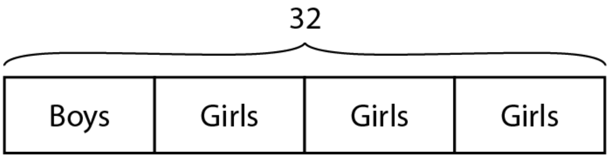
3. Comparison (additive)

Bar models can be used effectively to represent problems involving any sort of additive comparison; for example: *'What's the difference between…?'* or *'How many more to make…?'*. Such diagrams support the idea of subtraction as difference and as the inverse of addition.

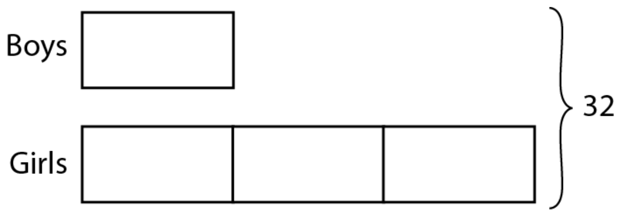


4. Comparison (multiplicative)

Problems involving situations where two or more quantities are connected by a multiplicative relationship or ratio can be succinctly modelled. For example, if we know that there are three times as many girls as boys in a class and there are 32 students altogether, the following bar model diagram can be drawn.



Alternatively, two bars could be drawn, with one bar representing the number of boys and the other representing the number of girls.



While this second way of representing the situation does very clearly show the ratio of boys to girls as 1 : 3 (and is, therefore, a powerful way of depicting any ratio $m : n$), using a bracket to label the total quantity, rather than having a bar that represents the 'whole', may be a new way of thinking for students.

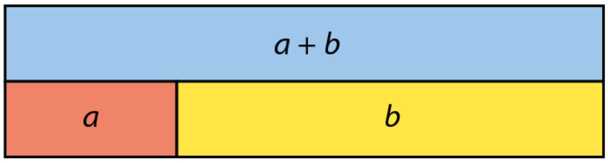
Prior to Key Stage 3, students will have used two bars in additive comparison structures; for example, when comparing a 'part' to a 'whole'. Seeing the total of the two bars as the 'whole' may not come naturally to some students, and they may find the 'whole' easier to identify when the bar itself represents the 'whole'. The need for a clear visual reminder of the 'whole' is important when using bar model diagrams, and students should explore different ways of depicting problems in order to identify the most accessible way in which to represent problems that make the embedded structures clearly visible.

Bar models provide students with a powerful tool for visualising and making sense of word problems and revealing the structure of a problem. Bar models can also support students in deciding which calculations they need to perform and help them understand what to do in order to get to a solution. This understanding of the *why* and the *how* is essential for students when it comes to fluently applying their skills and knowledge to a wide range of mathematical problems. Students need to have a good understanding of key concepts when using bar models and, once they are able to confidently use bar models as part of their reasoning process, they can apply the technique to many different areas of the mathematics curriculum.

Why they are important

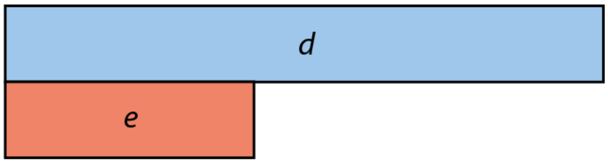
Bar models provide a bridge for students between concrete objects and abstract, symbolic representations. They are an ideal representation to support students in understanding key ideas in the secondary mathematics curriculum (for example, ratio, proportion and the idea of a variable). Bar models can support students in understanding key mathematical structures. However, if students have not understood additive structure, they may not initially find that bar models help them. There is a subtle interplay between coming to understand a mathematical structure and being able to use a representation, like the bar model. For representations to become useful for students, they need to be offered consistently over time and in different contexts.

A bar model depicting an additive structure can be used to discuss the idea of compensating addition.

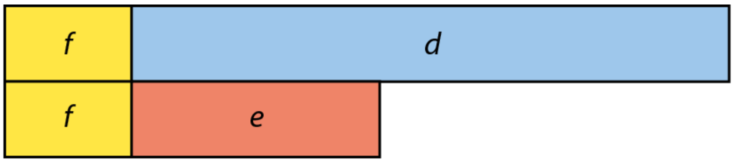


If the total remains constant and one addend is increased, the other addend must **decrease by the same amount**. For example: $237 + 199 = 236 + 200$; $2.89 + 5.27 = 3 + 5.16$; [and generally] $a + b = (a + c) + (b - c)$.

When considering subtraction as difference, the idea of equal subtractions can be examined.

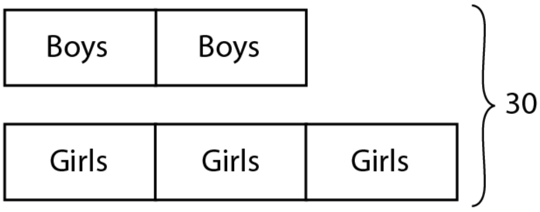


If both the minuend and the subtrahend are increased by the same amount, the difference remains the same.

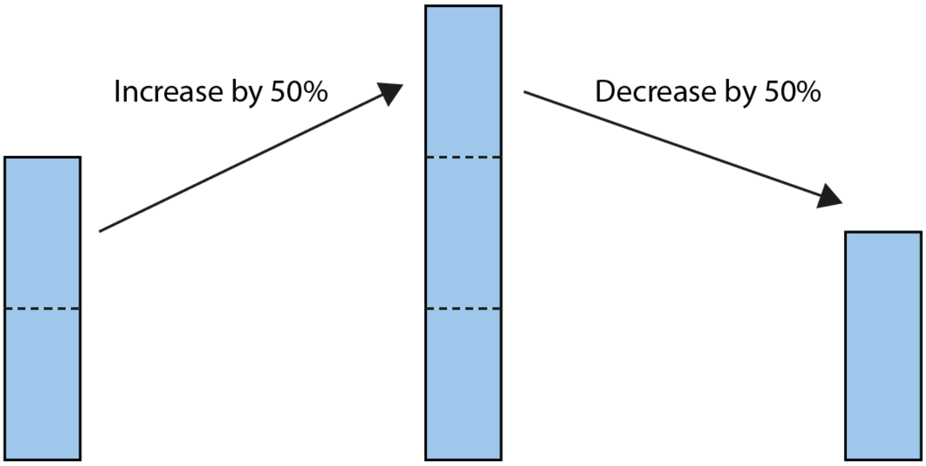


For example: $5413 - 1977 = 5436 - 2000$; $2.01 - 0.92 = 2.09 - 1$; [and generally] $d - e = (d + f) - (e + f)$.

Bar models are powerful images to both represent and understand ratio. In the example below, of a class of 30, with a 2 : 3 ratio of boys to girls, it can clearly be seen that $\frac{2}{5}$ of the class are boys and $\frac{3}{5}$ are girls.



Bar models can be used to examine and explore percentage increase and decrease, and the idea of a single multiplier. For example, the problem, *'When you increase a quantity by 50% and then decrease the new amount by 50%, why don't you get back to the original amount?'*, can be represented using a bar model diagram.

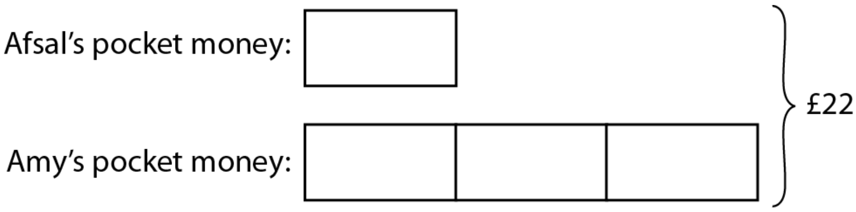


The idea of an increase of 50% being equivalent to multiplying by 1.5 can then be introduced.

By drawing bars for unknown quantities, the idea of a variable can be introduced.

Example problem:

Amy and Afsal both get weekly pocket money. Amy receives three times as much pocket money as Afsal, and between them they receive £22.



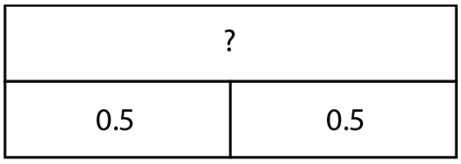
The amount each child receives can be modelled using a bar model diagram, and the idea of writing Afsal's pocket money as x and Amy's as $3x$, resulting in the equation $x + 3x = 22$, can be discussed.

Students often struggle, when faced with a word problem in mathematics, to get a sense of the mathematical structure of the problem, extract the relevant information and devise a solution. In problem-solving, it is often not doing the mathematics that is the stumbling block, but knowing what mathematics to do. Bar models support this as they provide a way of revealing structures and relationships within the problem.

How they might be used

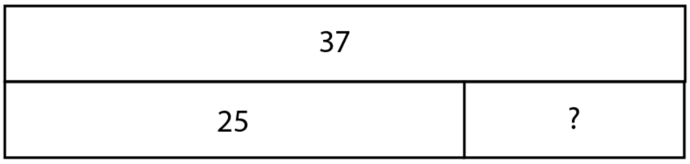
Problems involving the four arithmetic operations

Bar models can not only be used as a representation of the operations of addition, subtraction, multiplication and division, but can also highlight the relationship between them. They support students in representing problems pictorially to reveal additive and multiplicative structures. Addition and multiplication problems can be thought of as 'whole unknown' problems, whereas subtraction problems are 'part unknown' problems.



Additive: $? = 0.5 + 0.5$

Multiplicative: $? = 2 \times 0.5$



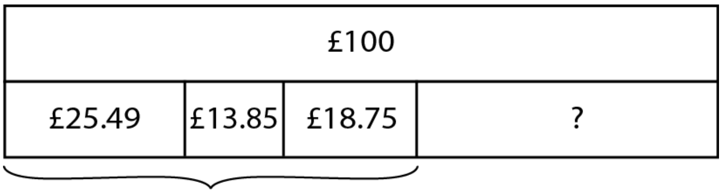
$37 = 25 + \mathbf{?}$

$37 - \mathbf{?} = 25$

$37 - 25 = \mathbf{?}$

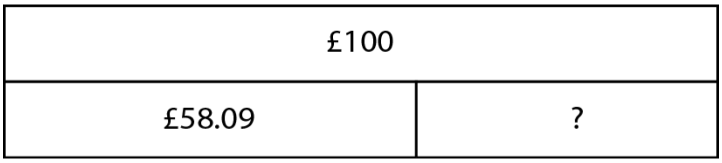
The 'part unknown' bar model supports students in making the association between the operation of subtraction and finding the missing part. When there is more than one subtrahend, a bar model can also help students to recognise that when subtracting more than one quantity, adding the subtrahends first, and then subtracting the resulting total from the minuend, is equivalent to (and more efficient than) repeatedly subtracting each subtrahend.

For example, if a school raises £100 to spend on a party and spends £25.49 on food, £13.85 on drinks and £18.75 on decorations, a bar model can be drawn to help to identify strategies for finding the amount left over and support students in deciding which is the most efficient strategy.



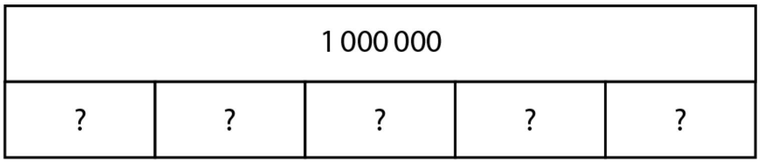
$$\begin{array}{cccccc} & £ & 2 & 5 & . & 4 & 9 \\ & £ & 1 & 3 & . & 8 & 5 \\ + & £ & 1 & 8 & . & 7 & 5 \\ \hline & \mathbf{£} & \mathbf{5} & \mathbf{8} & \mathbf{.} & \mathbf{0} & \mathbf{9} \\ \hline & & 1 & 2 & & 1 & \end{array}$$

When using repeated subtraction, it may be helpful to draw a revised bar model diagram after each subtraction, to make each stage of the calculation process clear. It is also important for students to recognise that, unlike addition problems with several addends, a single-column calculation cannot be used to subtract multiple subtrahends.

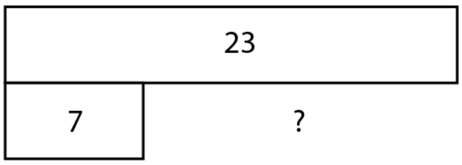


Revising the bar model with the total of the three subtrahends reveals the simplified structure of the problem and the single amount to be subtracted from the minuend of £100.

Division problems can either be 'size of part unknown' or 'number of parts unknown'. When the size of part is unknown, but the number of parts is known, a bar model diagram with equal parts can be drawn as, for example, the division of one million into five equal parts.

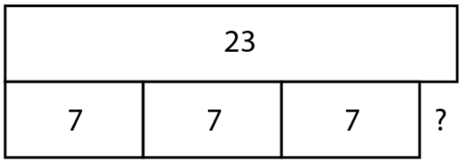


When drawing a bar model to represent a 'number of parts unknown' problem, one part can be drawn and the diagram used to establish the calculation required to find the total number of parts that equal the whole. For example, Asha has 23 vitamins and will take one vitamin a day. She wants to know how many weeks' supply of vitamins she has left.

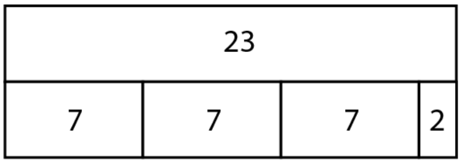


(Note the use of a question mark, rather than a bracket, to prevent confusion with a subtraction.)

The bar diagram can be used to emphasise the link between skip counting (i.e. counting up in 7s) to find the number of parts and modelling the problem as a division.



The difference between the top and bottom bars represents the remainder when dividing 23 by seven.

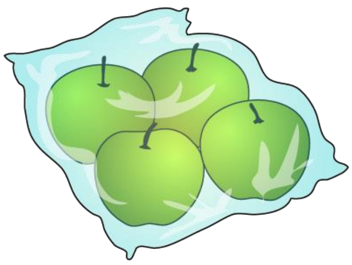


Identifying the remainder as two and understanding that this represents the number of vitamins left over after three weeks' supply, is a key element in interpreting the problem. The bar model diagram supports students' understanding that the two must be related additively because it is not the same size as the 7s – this is an application of a generalisation, recognised at primary school, that the remainder is always less than the divisor.

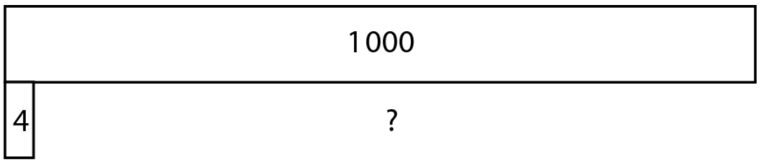
While bar models can be used to depict division as both grouping (quotitive) and sharing (partitive), the numbers involved in the problem may influence the way that students represent the problem.

Example problem:

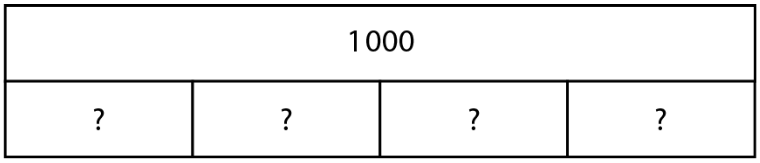
1 000 apples are packaged into packets of four apples. How many packets will there be?



Here, the total quantity of apples and the size of group (number of apples in a packet) are known (quotitive) and so can be represented as a 'grouping' problem.



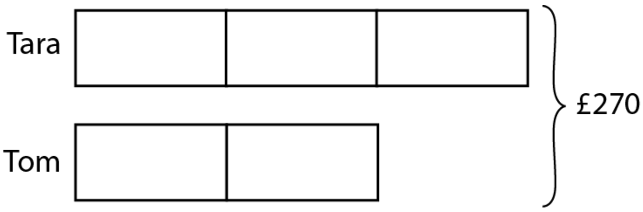
However, grouping 1 000 items into groups of four may not be as efficient a strategy as taking a partitive approach and sharing the total number of apples by four (the number of apples in a packet) to get the total number of packets.



Once students have a deep understanding of the structure, they can decide on the most efficient way of representing the problem, which may be rooted in a different structure to the one they used to make sense of the problem.

Modelling additive and multiplicative relationships

The comparison bar model is a particularly helpful representation when it comes to ratio problems. For example, if we want to work out how much money Tom and Tara receive if they share £270 between them in the ratio 2 : 3, we can represent the problem with a bar model.

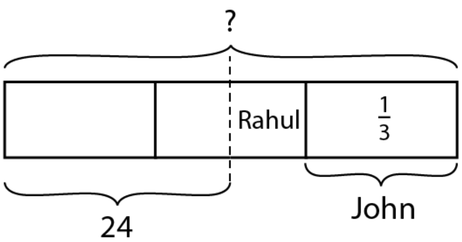


The comparison bar model makes it easy to see that the whole amount is made up of five parts (2 + 3) and so the £270 needs to be first divided by five, before being multiplied by the number of parts each person gets. Similarly, ratio problems when the total is not known, but one of the smaller amounts or even the difference between the amounts is known, can be more easily recognised and tackled by students when a bar model is drawn.

Bar models offer an accessible method of representing one way of understanding fractions. Once students have an appreciation of the bar as a whole divided into equal pieces, and that the number of pieces the bar is divided into can represent the denominator, there are a wide range of contexts for which a bar model can be used to represent fractions problems.

Example problem:

Susan has a bag of sweets. She gives one-third of them to John and then one-quarter of the remaining sweets to Rahul. If Susan has 24 sweets left in the bag, how many sweets were in the bag to start with?

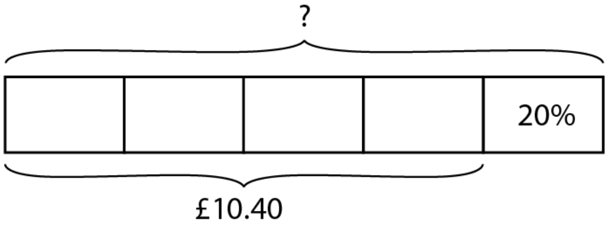


Drawing a bar model diagram to represent the problem reveals the fact that $\left(\frac{1}{4}\times\frac{2}{3}\right)+\frac{1}{3}=\frac{1}{2}$, without the need for finding equivalent fractions with a denominator of 12 and performing a long and intricate calculation. The need to double the number of sweets left in the bag (24) to find the number of sweets in the bag at the start is also made clear, showing how the bar model can reveal the mathematical structure within a problem and provide students with an insight into how to solve it.

Finding the original cost of an item that has been reduced in a sale is a problem type that students can struggle with. They often make the incorrect assumption that if they find the sale percentage of the reduced price and add it back on, they will get back to the starting price.

Example problem:

A DVD was reduced in a sale by 20% and now costs £10.40. What was the original price?



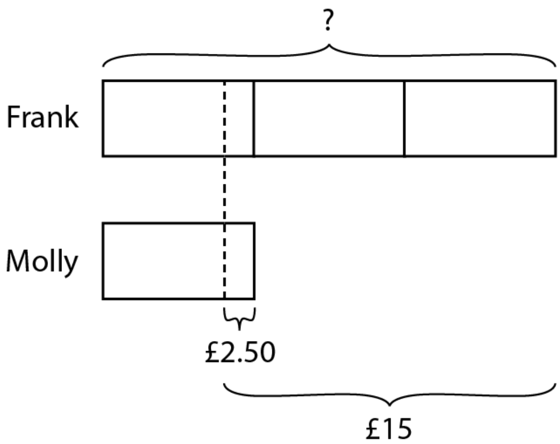
Problems involving percentages can be solved in the same way as those involving fractions, i.e. by dividing the whole into equal parts. Recognising that 20% is equivalent to $\frac{1}{5}$ is a key concept here and, once students have recognised the need to divide the whole into five parts, the process of dividing the sale price by four (to find one part) and then multiplying by five to find the whole, can be identified.

The ease with which the whole can be identified, and so allowing students to decide whether the whole or only part of the whole is known, makes bar models a helpful representation for problems like this.

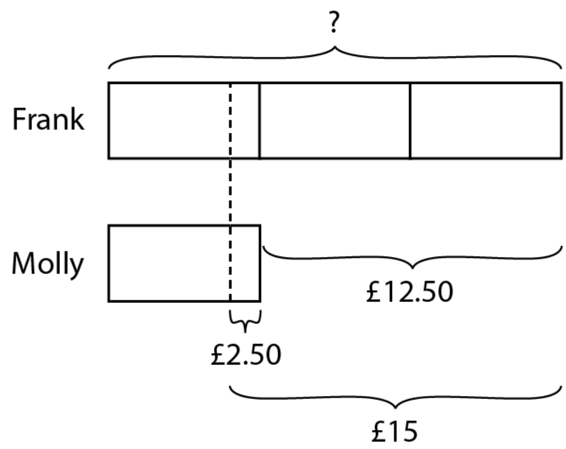
Problems involving money provide opportunities for students to calculate with decimals, and bar model diagrams can help students to identify the calculations needed.

Example problem:

Frank has three times as much money as Molly. After Frank spends £15 and Molly spends £2.50, they each have an equal amount of money. How much money did Frank have originally?



The initial setting up of the bar model representation for this problem is fairly straightforward, as we know that Frank had three times as much money as Molly, so the bar representing the amount of money Frank had needs to be three times the bar representing Molly's money. Recognising that the amount of money spent by Frank must be represented by more than two of the three bars, to coincide with the fact that Molly has spent some money too, may be less obvious to students. However, labelling the £2.50 spent by Molly before labelling the £15 spent by Frank makes this more apparent. Once these amounts have been represented correctly, identifying the difference between the amounts spent as the total of two of the parts, provides an insight into the strategy for solving the problem.

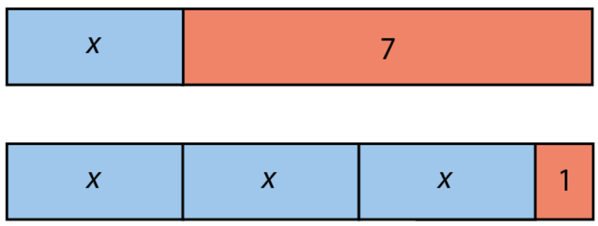


Once the value of one part has been identified as £6.25, the total money Frank had to start with can be found, either by multiplying one part by three, or by adding one part to the two parts total already known (£12.50). The bar model makes both of these methods visible and demonstrates again the way in which bar model diagrams reveal the structure of a problem and allow students to see the calculation method needed when finding a solution.

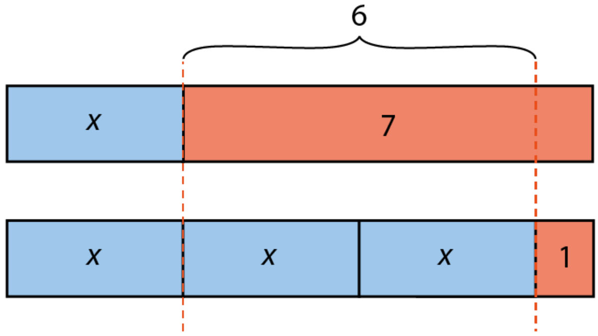
Solving linear equations

While algebra tiles provide students with physical manipulatives that can be picked up and moved around, the concept of equality is shown less well using this representation. Bar model diagrams, in contrast, can depict equality between two expressions very well and, one could argue, in a way that reveals the equation's structure more readily. This, however, can become problematic when $x < 0$, and so the decision of when to use bar models to support students in understanding solutions to linear equations should be carefully considered. It is helpful to examine the symbolic algebra alongside the bar model diagrams in order to relate the manipulation of the bars with the manipulation of the symbols, so that the bar model is used as a bridge to fluent use of symbols, rather than instead of. In this way, when bar models break down, the symbols can take over.

For example, the equation $x + 7 = 3x + 1$ can be represented using two bars of equal length: one representing the left-hand side of the equation and the other the right-hand side.

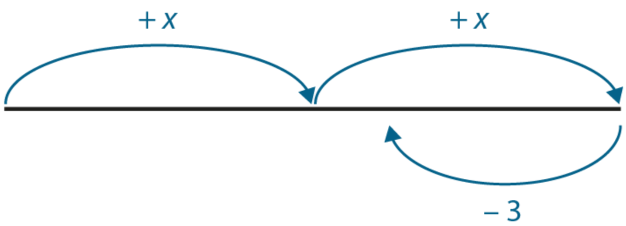


Equality is clearly shown by lining up the two bars, and this depiction leads to the intuitive strategy of 'cutting off' equal sections at either or both ends in order to simplify the equation.

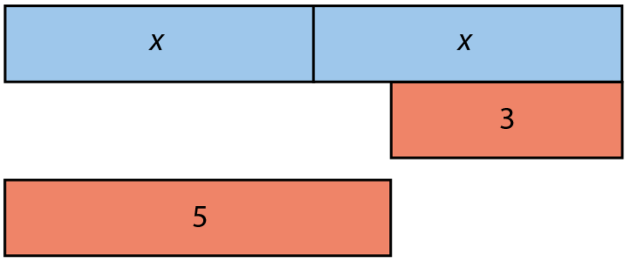


This can be done in stages to give $7 = 2x + 1$ (and so $6 = 2x$) or $x + 6 = 3x$ (and so $6 = 2x$). Alternatively, by cutting off the equal sections at each end simultaneously, the equation $6 = 2x$ (or $2x = 6$) can be identified directly, and with a final division line between the two bars each representing x, the solution of $x = 3$ is clearly represented.

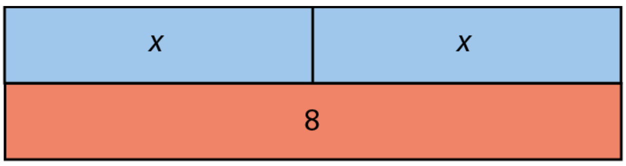
When solving equations containing a negative term, e.g. $2x - 3 = 5$, the bar model diagram is derived from an understanding of the structure of the calculation on the number line and the difference model.

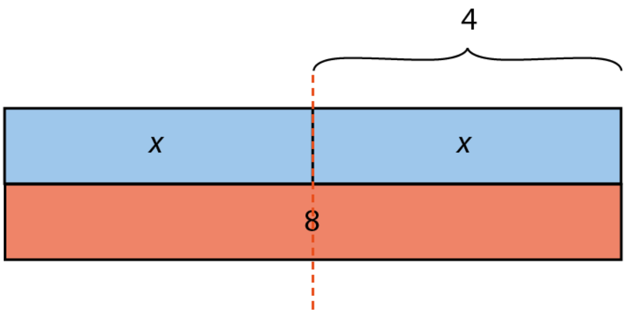


When drawing bar model diagrams for each side of the equation, it is important to recognise that the rectangle representing five is the same length (due to the equality) as the difference between the two bars that represent $2x$ and the bar representing three.

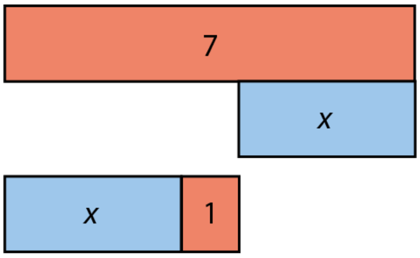


The bar model 'slides' together to form the equation $2x = 8$, from which a simple division by two gives a solution of $x = 4$.

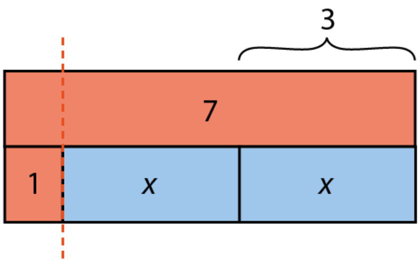




The same method can be used when an algebraic term is negative. For example, the equation $7 - x = x + 1$ can be drawn as:

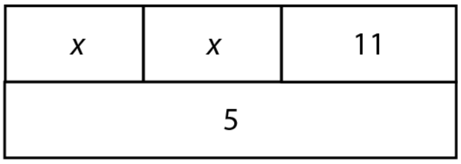


Sliding the bars together reveals the equation $7 = 2x + 1$. The next diagram highlights the rearrangement of the bars as representing the collection of like terms to change $x + 1 + x$ into $1 + 2x$.



Cutting 1 off the end of the bar model diagram to leave $6 = 2x$, and then dividing the remainder by two, shows the solution of $x = 3$.

The limitations of bar model diagrams become apparent when representing equations with negative solutions, as no sensible diagram can be drawn. For example, if we draw a bar model diagram to represent the equation $2x + 11 = 5$, the '5' bar has to be much bigger than the '11' bar.



While in some ways this can be regarded as a clear indicator that x has a negative value, if students do not recognise this, then it can be hard for them to know where to begin. Use of the bar in positive solutions, however, should provide a tool for supporting students in making generalisations. These generalisations can then be tested out in other contexts (for example, where the solution is negative) and students can see that the generalisations hold true. As the representation is there to reveal structure and not to 'get the answer', the use of the bar model alongside the algebraic symbols supports students in developing a deep understanding of additive and multiplicative relationships and their combined use when solving linear equations.

Further resources

When using bar model diagrams to reveal the structure of an equation that needs solving, there is often the need to redraw the bar model as the solution process progresses. The ability to quickly reproduce or amend a bar model diagram, without the need for laborious hand-drawn diagrams, can often be beneficial.

See, for example:

Math Playground
https://www.mathplayground.com/thinking_blocks_modeling_tool/index.html
Bar model diagrams can be generated, manipulated and annotated using this interactive modelling tool. Bars can be dragged into the work area and the width of the bar changed as desired, by dragging the left or right side of the bar (minimum bar width of two grid boxes). Bars can be duplicated using the copy button, divided into equal parts with the slider, and then these parts can be ungrouped or joined back together. Selecting the scissors button allows the bar to be cut into two equal parts. Brackets can be added to the work area and text added to label what the bars represent. There is also the option to change the appearance of the bar model diagram using the colour palette, to select more than one bar (to allow multiple bars to be moved around the work area at the same time) and to toggle the grid background on and off. A single object can be deleted by selecting it and then clicking the trash button. The refresh button clears the work area.

The Mathenæum: modelling word problems
http://thewessens.net/ClassroomApps/subindex.html?id=wordproblems&topic=models&path=Models
A suite of additive and multiplicative word problems to model and solve using interactive bar models. There is a choice of 'Part–Whole' or 'Comparison' bar modelling for both the additive and multiplicative problems, with the option to select a level of difficulty ranging from 1 to 4. A hint can also be given. Once the problem has been correctly represented with a bar model, by dragging the blue divide and end lines, the solution can be calculated and entered. If students are using these problems as part of their independent work, it is important to check that solutions are being obtained as a result of a deep understanding and not merely as a result of trial and error.

The Mathenæum: fractions, decimals and percentages
http://thewessens.net/ClassroomApps/subindex.html?id=fractions&topic=models&path=Models
More word problems, in the same format as those described above, using bar models to visualise fractions, decimals, percentages and ratio problems, with the choice of the following types of problem:

- Visualising Fractions
- Comparing Fractions
- Adding and Subtracting Fractions
- Multiplying Fractions
- Dividing Fractions
- Decimals
- Percentages
- Ratios.

Once the problem type has been selected, there are further choices on the kind of problem to attempt, with an added free-play 'Explore' feature within the 'Visualising Fractions', 'Comparing Fractions' and 'Multiplying Fractions' problem sets.

With any online interactive problem, especially ones like those described above, where feedback is given when the correct solution is obtained, it is important that students' focus continues to be on deepening their mathematical understanding and is not merely about completing a performance chart. It may be more appropriate to use activities such as these in a whole-class or small-group setting, so that students' exploration of the problems, with the aid of bar model diagrams, can be fully supported.

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